10 Orthonormal bases

(10.1) Definition *A list $X_1, \ldots, X_k$ of points (or column vectors) in $\mathbb{R}^n$ is* orthonormal *if*

$$X_i^T X_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A square matrix S is orthogonal *if*

$$S^{-1} = S^T.$$

(10.2) Proposition (i) *An orthonormal list $X_1, \ldots, X_n$ in $\mathbb{R}^n$ is a basis.*
(ii) *Given this orthonormal basis, the change-of-basis matrix*

$$S = [X_1 \ \ldots \ X_n]$$

is orthogonal. (Easy proofs omitted.) ∎

For example, $(3/5, 4/5)$ and $(-4/5, 3/5)$ form an orthonormal basis for $\mathbb{R}^2$.

It may be useful to construct orthonormal bases in 2 and 3 dimensions.

A 2-dimensional example. Find an orthonormal basis with one axis aligned along OW_1 where $W_1 = (1, 2)$.

'Normalise' W_1, i.e., divide it by its length (as a vector, distance from O as a point), to get X_1.

$$X_1 = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$

X_2 should be a unit-length vector orthogonal to X_1. There are two possibilities. The positive or negative normal to X_1. Either will produce an orthonormal basis, but we choose the positive normal.

$$X_2 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right),$$

$$S = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Convert $[3, 4]^T$ from standard to new coordinates. That is, multiply by S^{-1}. But $S^{-1} = S^T$.

$$S^T \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{11}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$$

If we multiply by S we recover the standard coordinates:

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{11}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{15}{5} \\ \frac{20}{5} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(10.3) Definition *To* normalise *a vector means to divide it by its length, producing a vector of unit length.*

A 3-dimensional example. Find a right-handed orthonormal basis in which the third axis is in the direction $W_3 = (1,1,1)$.

This time we start with X_3, which is $(1,1,1)$ normalised

$$X_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

First find a point on an axis perpendicular to OW_3. (We postpone normalisation till later.) Take the cross-product with another point: $\vec{k} = (0,0,1)$ will work here, because it is not in the same direction as W_3.

By taking cross-product with W_3 rather than X_3, we postpone the need for square roots.

$$V_1 = (1,1,1) \times (0,0,1) = (1,-1,0)$$

With normalisation still postponed, let $V_2 = (1,1,1) \times V_1$. This time the order matters:

$$V_1 = W_3 \times \vec{k}, \quad V_2 = W_3 \times V_1,$$

If we took $V_1 \times W_3$ we would get a left-handed system.

$$V_2 = (1,1,1) \times (1,-1,0) = (1,1,-2)$$

Finally, normalise and use column vectors

$$X_1 = V_1/\sqrt{2}, \quad X_2 = V_2/\sqrt{6}, \quad X_3 = W_3/\sqrt{3}$$

$$[X_1, X_2, X_3] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Find the coordinates of $(2,3,4)$ with respect to this basis.

$$S^T \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{3}{\sqrt{6}} \\ \frac{9}{\sqrt{3}} \end{bmatrix}$$