19 Normal distribution

The Normal distribution. The Normal or Gaussian distribution is the limiting case of *any halfway reasonable distribution.* We generally expect it to apply when we are taking measurements, etcetera.

Recall the distribution is based on the PDF (probability density function)

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

The 'standard' form is where $\mu = 0$ and $\sigma = 1$:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

There is no 'closed form' for its integral. You need to use statistical tables.[1]

19.1 Normal distribution tables

It can be shown (i) that this probability density function integrates to 1, as is necessary; (ii) the mean is μ and (iii) the standard deviation is σ, and if $X \sim N(\mu, \sigma^2)$ then

$$\text{(iv)} \quad \frac{X-\mu}{\sigma} \sim N(0,1).$$

The tables are given for $x \geqslant 0$. At $x = 0$ the probability is 1/2, the PDF being symmetric.

For brevity, we write $P(x)$ for $\text{Prob}(X \leqslant x)$ (under the $N(0,1)$ distribution). If $x \geqslant 0$ this is tabulated (for some x).

- If $0 \leqslant a < b$, $\text{Prob}(a \leqslant X \leqslant b) = P(b) - P(a)$.
- If $a < 0$ then $P(a) = 1 - P(-a)$.
- If $a < 0$ and $b > 0$ then $\text{Prob}(a \leqslant X \leqslant b) = P(b) + P(-a) - 1$.
- If $a < b < 0$ then $\text{Prob}(a \leqslant X \leqslant b) = P(-a) - P(-b)$.

19.2 Working with the normal distribution.

Example. Say $X \sim N(0,1)$. Meaning: X is a random variable distributed $N(0,1)$. We use 'Prob' with its obvious meaning. Calculate $\text{Prob}(X \leqslant 0.3))$.

Answer: directly from the official tables, .6179.

Calculate $\text{Prob}(-0.2 \leqslant X \leqslant 0.3)$.

Answer.

$$\begin{gathered}\text{Prob}(-0.2 \leqslant X \leqslant 0.3) = \\ \text{Prob}(X \leqslant 0.3) + \text{Prob}(X \leqslant 0.2) - 1 = \\ .6179 + .5793 - 1 = .1972\end{gathered}$$

[1] The official tables (revised in recent years) contain all that is needed for this course. Because of covid (2021), a few pages are posted on the module web page.

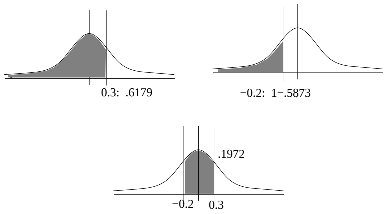

Figure 1: Prob $-0.2 \leqslant X \leqslant 0.3$

Calculate Prob$(-0.3 \leqslant X \leqslant -0.2)$:

Answer.

$$\text{Prob}(0.2 \leqslant X \leqslant 0.3) = .6179 - .5793 = .0386$$

Example. It is known that measuring a certain quantity follows a normal distribution with mean μ and variance σ^2.

Suppose for argument's sake that the variance σ^2 is known to be 1/100.[2]

Let X be the result of a measurement. Calculate the following probabilities.

$$\begin{array}{rl} \text{(i)} & \text{Prob}(X < \mu + 0.2) \\ \text{(ii)} & \text{Prob}(|X - \mu| < 0.2) \end{array}$$

Answer. The important fact is that when $X \sim N(\mu, \sigma^2)$,

$$\frac{X - \mu}{\sigma} \sim N(0, 1).$$

The mean μ is not given, but $\sigma = \sqrt{\sigma^2} = 0.1$.

(i)

$$X < \mu + 0.2 \iff \frac{X - \mu}{0.1} < 2 :$$

probability 0.9772.

(ii)

$$|X - \mu| < 0.2 \iff \left| \frac{X - \mu}{0.1} \right| < 2$$

$$\iff \text{above} \geqslant -2 \text{ and } \leqslant 2$$

$$.9772 - (1 - .9772) = .9544.$$

[2]It sounds unlikely that the variance could be known better than the mean.