15 Differential equations

15.1 Theory

The differential equation

$$\frac{dx}{dt} = \lambda x$$

has the solution

$$x = x_0 e^{\lambda t}.$$

We can use eigenvector methods to solve *simultaneous* differential equations

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy.$$

This system of ODEs can be expressed in matrix form

$$\left[\begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array}\right] = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right], \quad \text{or}$$

$$\frac{dX}{dt} = AX.$$

If A has real eigenvalues and eigenvectors, then we can write

$$A = SA'S^{-1}, \quad \text{where}$$

$$A' = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right].$$

Now the principle that when a is a constant, then $d(ax)/dt = adx/dt$, extends to systems of differential equations, and

$$\frac{d}{dt}S^{-1}X = S^{-1}\frac{dX}{dt} =$$

$$S^{-1}AX = S^{-1}SA'S^{-1}X = A'S^{-1}X.$$

Now S is a change-of-basis matrix, and $S^{-1}X$ gives the new coordinates, call them α and β, of X.

$$\frac{d}{dt}\left[\begin{array}{c} \alpha \\ \beta \end{array}\right] = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right]\left[\begin{array}{c} \alpha \\ \beta \end{array}\right].$$

This gives two independent differential equations

$$\frac{d\alpha}{dt} = \lambda_1 \alpha, \quad \frac{d\beta}{dt} = \lambda_2 \beta$$

The solutions are

$$\alpha = \alpha_0 e^{\lambda_1 t}, \quad \beta = \beta_0 e^{\lambda_2} t.$$

The solution involves two 'constants of integration.' They can be resolved if one is given the initial value of X: that at time $t = t_0$, $X = X_0$. Then

$$\left[\begin{array}{c} \alpha_0 \\ \beta_0 \end{array}\right] = S^{-1}X_0.$$

and

$$X = S \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = S \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} S^{-1} X_0 = e^{At} X_0.$$

15.2 Practice

15.2.1 Exponential of a matrix

$$e^{At} \quad \text{where} \quad A = \begin{bmatrix} -8 & 6 \\ -15 & 11 \end{bmatrix}$$

Changing to an eigenvector basis

$$A = SA'S^{-1} \quad = \quad \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

the solution is

$$e^{At} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2e^t & 3e^{2t} \\ 3e^t & 5e^{2t} \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 10e^t - 9e^{2t} & -6e^t + 6e^{2t} \\ 15e^t - 15e^{2t} & -9e^t + 10e^{2t} \end{bmatrix}$$

15.2.2 Sample ODE

Solve the system of ordinary differential equations

$$dx/dt = -8x + 6y$$
$$dy/dt = -15x + 11y$$

subject to $x = 7$ and $y = 3$ when $t = 0$.

Answer.

$$\begin{bmatrix} 10e^t - 9e^{2t} & -6e^t + 6e^{2t} \\ 15e^t - 15e^{2t} & -9e^t + 10e^{2t} \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad = \quad \begin{bmatrix} 52e^t - 45e^{2t} \\ 78e^t - 75e^{2t} \end{bmatrix}.$$

How can this be checked? First, check the value at $t = 0$: yes, it equals $[7\ 3]^T$. Next, evaluate

$$\begin{bmatrix} -8 & 6 \\ -15 & 11 \end{bmatrix} \begin{bmatrix} 52e^t - 45e^{2t} \\ 78e^t - 75e^{2t} \end{bmatrix} : \quad \begin{bmatrix} 52e^t - 90e^{2t} \\ 78e^t - 150e^2 t \end{bmatrix}$$

Then evaluate

$$\frac{d}{dt} \begin{bmatrix} 52e^t - 45e^{2t} \\ 78e^t - 75e^{2t} \end{bmatrix} : \quad \begin{bmatrix} 52e^t - 90e^{2t} \\ 78e^t - 150e^2 t \end{bmatrix}$$

So,

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

as required.

15.3 More examples

Calculate the eigenvalues of

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda - 1 \end{vmatrix} = 0$$
$$(\lambda - 1)^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$\lambda = 3, -1$$

With 2×2 matrices, it is usually easy to produce eigenvectors for given eigenvalues.

$$\lambda = 3 : \lambda I - A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$$

Take the *cofactors* of any row and transpose (unless that row is entirely zero) In the 2×2 case, the cofactors of a row are equivalent to the positive or negative normal of the *other* row.

Positive normal to $(-1, 2)$ is $(-2, -1)$. It is OK to scale an eigenvector, so to make it neater we use $(2, 1)$ instead (which becomes a column vector). You can check that

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} =$$

as required.

$$\lambda = -1 : \lambda I - A = \begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix}$$

Positive normal to bottom row $(2, -1)$:

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ -1 \end{bmatrix} =$$

as required.

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix},$$

and

$$S = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

the change-of-basis matrix with inverse

$$S^{-1} = \frac{-1}{4} \begin{bmatrix} -1 & -2 \\ -1 & 2 \end{bmatrix},$$

and

$$A' = D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

(We write D as a reminder that it is diagonal). As usual

$$A = SDS^{-1}$$

Another differential equation example.

$$\frac{dx}{dt} = x + 4y \qquad \frac{dy}{dt} = x + y;$$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Solution. Find eigenvalues and eigenvectors. We get (of course, the eigenvectors can be scaled by any amount):

$$\lambda = 3, 1$$

$$\lambda = 3 : \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda = -1 : \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \qquad A = SDS^{-1}, \quad \text{where}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Suppose we have an initial value condition as well: $(x, y) = (4, 1)$ when $t = 0$. Then

$$\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

so $\alpha_0 = 3/2$ and $\beta_0 = -1/2$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$$

(Needs to be multiplied out.)