6 More facts about determinants

6.1 Cramer's Rule

Cramer's rule generalises to solve $AX = B$ where $A_{n\times n}$ has nonzero determinant. Gauss-Jordan elimination is usually more efficient for $n \geq 4$, and we shall not look at any new examples.

6.2 Product rule.

(6.1) Theorem *Given* $A_{n\times n}, B_{n\times n}$

$$\det(AB) = \det A \det B.$$

Sketch proof. Recall that for every ERO (on A), there is an $n \times n$ *elementary matrix* E such that the matrix obtained from the ERO coincides with

$$EA$$

also, we could see by inspecting the three kinds of ERO, that

$$\det(EA) = \det(E)\det(A).$$

Now, given that F is the RREF of A, there exists a matrix Q, product of elementary matrices, such that

$$A = QF \quad \text{and} \det(AB) = \det(Q)\det(FB).$$

Either

- $F = I$, $A = Q$, and $\det(AB) = \det A \det B$, or
- The bottom row of F is zero, $\det(A) = 0$, and $\det(FB) = 0$, since the bottom row of FB is zero, so

$$\det(AB) = \det(Q)\det(FB) = 0$$
$$\det(A) = 0 \quad \text{so}$$
$$\det(AB) = \det(A)\det(B). \quad \blacksquare$$

6.3 Invertibility

Consider again $A = QF$ as above. Recall

$$\det(A) = \det(Q)\det(F)$$

and note that $\det(Q) \neq 0$ (by analysis of EROs).

- If A is invertible, then $F = I$ and $\det(A) \neq 0$.
- If A is not invertible, then the bottom row of F is zero, and $\det(F) = 0$, so $\det(A) = 0$. $\blacksquare$

That is

(6.2) Theorem $\det(A) \neq 0 \iff$ *A is invertible.*

6.4 Determinant of transpose

(6.3) Theorem *Given* $A_{n\times n}$*,*

$$\det(A^T) = \det(A).$$

Sketch proof. Write

$$A = E_k E_{k-1} \cdots E_1 F$$

as previously, verify the result for the elementary matrices E_j and the RREF F, and observe

$$A^T = F^T E_1^T \cdots E_{k-1}^T E_k^T. \quad \blacksquare$$

From this result, it follows that all the properties relating determinant to the rows of a matrix apply to the columns of the matrix:

- Cofactor expansion along any column.
- If any column is zero, the determinant is zero (already proven).
- If two different columns are swapped, the sign of the determinant is reversed.
- If two different columns are equal, the determinant is zero.
- Multilinearity in terms of columns.