12 Linear maps and matrices

Every linear map f can be encoded by a matrix A, in the sense that if a point is represented by a column vector X then AX represents $f(X)$. Examples:

Rotation through ϕ around O in $\mathbb{R}^2$:

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Perpendicular projection onto the z-axis:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perpendicular projection onto the xy-plane

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Perpendicular reflection thrrough the xy-plane:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Rotation around the z-axis:

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The relation between maps and matrices is: Let $E_1, E_2, \ldots, E_n$ be the standard basis for $\mathbb{R}^n$. Then the first column of A represents $f(E_1)$, the second $f(E_2)$, and so on.

Verify this with the above matrices.

We shall construct matrices for perpendicular projection onto lines and planes, and rotations about given axes, and possibly (perpendicular) reflections.

These lectures have already given formulae for applying projections, reflections, and rotations to any point, and they can be used to construct the matrices if one wants.

But there is another, more general, approach: find coordinate axes in which the map takes a simple, or at least a recognisable, form, and convert to standard coordinates. We do this only in $\mathbb{R}^3$.

(12.1) Definition *Let X_1, X_2, X_3 be an ordered basis defining a 'new' coordinate system, and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map.*

Then the matrix A' *of* f in new coordinates *is the* 3×3 *matrix whose first, second, and third columns are the* new *coordinates of $f(X_1), f(X_2)$, and $f(X_3)$, respectively.*

(12.2) Lemma Fact. *If A' is the matrix of a linear map f in new coordinates, and $N = [\alpha\ \beta\ \gamma]^T$ are the new coordinates of a point P, then $A'N$ are the* new *coordinates of $f(P)$.* ∎

(12.3) Corollary *Given an ordered basis X_1, X_2, X_3, forming the columns of a 'change of basis' matrix S, f is a linear map with matrix $A_{3\times 3}$ in standard coordinates and $A'_{3\times 3}$ in new coordinates, then*[1]

$$A = SA'S^{-1}. \quad \blacksquare$$

For rotation through 60° around $(1,1,1)$:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} =$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} =$$

$$\begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix}.$$

Example. Give the matrix for rotating 45° around the axis through $(12, 16, 99)$.[2]

```
  12   16   99   12   16    99
x             1    4   -3     0
--------------    -------------
  16  -12    0  297  396  -100   normalised as shown
```

Continuing: note, $1/2\sqrt{2}$ means $(1/2)\sqrt{2}$, i.e., $\sqrt{2}/2$ i.e., $1/\sqrt{2}$.

$$S = \begin{bmatrix} 4/5 & \frac{297}{505} & \frac{12}{101} \\ -3/5 & \frac{396}{505} & \frac{16}{101} \\ 0 & -\frac{20}{101} & \frac{99}{101} \end{bmatrix}, \quad A' = \begin{bmatrix} 1/2\sqrt{2} & -1/2\sqrt{2} & 0 \\ 1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^{-1} = S^T = \begin{bmatrix} 4/5 & -3/5 & 0 \\ \frac{297}{505} & \frac{396}{505} & -\frac{20}{101} \\ \frac{12}{101} & \frac{16}{101} & \frac{99}{101} \end{bmatrix},$$

$$A = SAS^{-1} = \begin{bmatrix} \frac{10057}{20402}\sqrt{2} + \frac{144}{10201} & -\frac{10191}{20402}\sqrt{2} + \frac{192}{10201} & \frac{1188}{10201} + \frac{214}{10201}\sqrt{2} \\ \frac{9807}{20402}\sqrt{2} + \frac{192}{10201} & \frac{9945}{20402}\sqrt{2} + \frac{256}{10201} & \frac{1584}{10201} - \frac{1398}{10201}\sqrt{2} \\ -\frac{1402}{10201}\sqrt{2} + \frac{1188}{10201} & -\frac{186}{10201}\sqrt{2} + \frac{1584}{10201} & \frac{9801}{10201} + \frac{200}{10201}\sqrt{2} \end{bmatrix}$$

Example Find the matrix for rotating points through around the axis through $(1, 1, 0)$.

```
  1  1  0      1  1  0   matrix 1  0  1  before
x        1   x 1 -1  0         -1  0  1  normalising
  -------      -------          0 -2  0
  1 -1  0      0  0 -2
```

[1]Caution: some texts may swap S and S^{-1}.

[2]This example shows a very laborious calculation, performed with computer algebra software. To calculate rotation matrices is usually very difficult.

$$S = \begin{bmatrix} 1/2\sqrt{2} & 0 & 1/2\sqrt{2} \\ -1/2\sqrt{2} & 0 & 1/2\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 1/2\sqrt{2} & -1/2\sqrt{2} & 0 \\ 1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S^T = \begin{bmatrix} 1/2\sqrt{2} & -1/2\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/2\sqrt{2} & 1/2\sqrt{2} & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1/4\sqrt{2}+1/2 & -1/4\sqrt{2}+1/2 & 1/2 \\ -1/4\sqrt{2}+1/2 & 1/4\sqrt{2}+1/2 & -1/2 \\ -1/2 & 1/2 & 1/2\sqrt{2} \end{bmatrix}$$

Example. One can use the $SA'S^T$ formula to get the matrix for projecting onto the plane $x + y + z = 0$.

Example. And projection onto the plane $x + y = 0$, using the basis from above.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Multiply by S^T:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$