2 Determinant in 2 and 3 dimensions

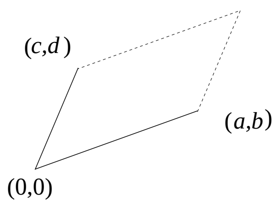
2.1 The determinant in 2 dimensions

(2.1) The determinant of points $P = (a, b)$ and $Q = (c, d)$ is
Notation.

$$\det(P, Q) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

The square arrangement $|\ldots|$ gives a single number, not a matrix.

(2.2) $\det(P, Q)$ is the signed area of the paralleogram $O, P, P + Q, Q$; positive, zero, or negative, depending on the angle POQ.



Example. Find the area of the triangle with corners $(1, 1), (2, 3), (3, 7)$.
Subtract $(1, 1)$ from the other two to get displacement vectors.
$(a, b) = (1, 2)$, $(c, d) = (2, 6)$.
The signed area of the parallelogram is $ad - bc = 1(6) - 2(2) = 2$. This is positive. The triangle has half the area, so the answer is 1.

(2.3) Geometrically, if $P = (a, b)$ then $(-b, a)$ is N_P, the positive normal to P (relative to the origin O), and [1]

$$\det(P, Q) = N_P \cdot Q = -P \cdot N_Q.$$

(2.4) Given a 2×2 matrix A, we define $\det(A)$ to be $P \cdot N_Q$ where P and Q are the rows of A. We could also let P and Q be the columns of A, because $\det(A) = \det(A^T)$.
Cramer's Rule. Cramer's Rule is a formula for solving linear systems.
To solve

$$ax + by = c$$
$$dx + ey = f$$

we let

$$P = \begin{bmatrix} a \\ d \end{bmatrix}, \quad Q = \begin{bmatrix} b \\ e \end{bmatrix}, \quad R = \begin{bmatrix} c \\ f \end{bmatrix}.$$

The equations become

$$Px + Qy = R.$$

Take the dot product with N_Q.

[1] $-P \cdot N_Q$ is a correction (sign) following 22/1/19

$$P \cdot N_Q x + Q \cdot N_Q y = R \cdot N_Q$$

Multiply by -1:

$$-P \cdot N_Q x - Q \cdot N_Q y = -R \cdot N_Q$$
$$\det(P, Q)x + \det(Q, Q)y = \det(R, Q).$$

Now, $\det(Q, Q) = 0$ no matter what Q is, so we get

$$\det(P, Q)x = \det(R, Q)$$
$$x = \frac{\det(R, Q)}{\det(P, Q)}$$
$$x = \begin{vmatrix} c & b \\ f & e \end{vmatrix} \div \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Similarly, $y = \det(P, R) \div \det(P, Q)$.

(2.5) For example, solve

$$x + 3y = 2$$
$$x + 7y = 3$$

Solution.

$$x = \frac{\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix}} = \frac{5}{4},$$
$$y = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix}} = \frac{1}{4}.$$

2.2 ADDITION: adjoint of 2×2 matrix

$$\operatorname{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

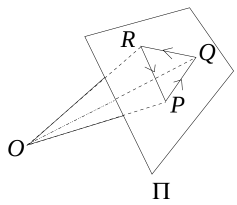
Fact: $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$. E.G.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(1)(4) - (2)(3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

2.3 Determinant in 3 dimensions

(2.6) Recall that the cross product $\vec{P} \times \vec{Q}$ is the unique vector $\vec{R}$ such that

- If $\vec{P}$ and $\vec{Q}$ are parallel then $\vec{R} = \vec{O}$. Otherwise,
- $\vec{R}$ is perpendicular to the parallelgram whose sides are $\vec{P}$ and $\vec{Q}$,
- The norm $|\vec{R}|$ is the area of that parallelogram, and
- $\vec{P}, \vec{Q}, \vec{R}$ form a right-handed system.
- The definition of right-handed system is as illustrated.



For P, Q, R to form a right-handed system,

  - The points PQR must be non-collinear,
  - the unique plane Π containing the triangle PQR cannot contain the origin O,
  - Given that O is 'below' the plane, then viewed from *above*, the triangle PQR has corners P, Q, R in *anticlockwise* order.
  - (Or clockwise when viewed from O).

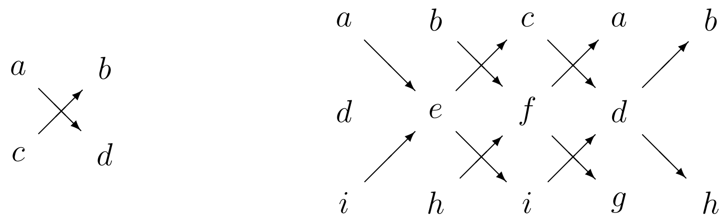
- And for calculation:

$$(a, b, c) \times (d, e, f) =$$
$$\left(\begin{vmatrix} b & c \\ e & f \end{vmatrix}, \quad -\begin{vmatrix} a & c \\ d & f \end{vmatrix}, \quad \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right)$$

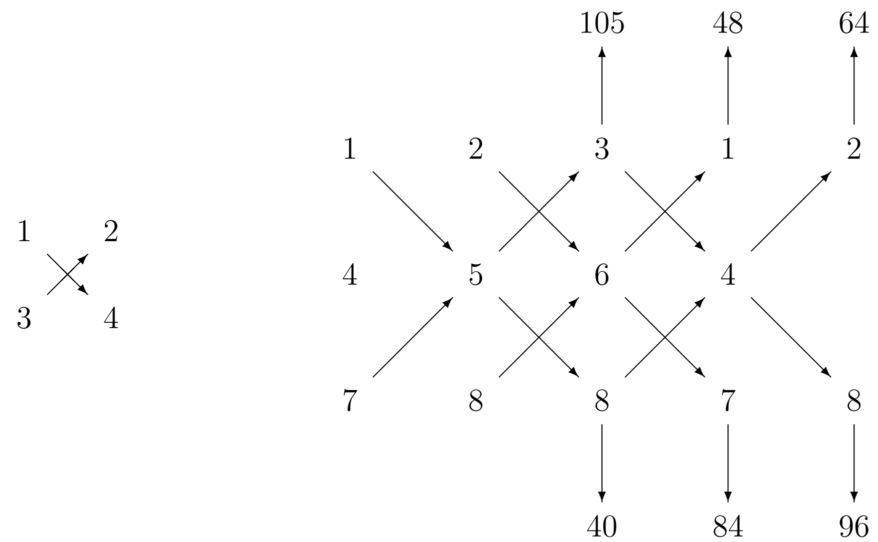
Important: the sign is different in the middle

(2.7) Definition $\det(P, Q, R) = P \cdot (Q \times R)$ *is called the* determinant *of* P, Q, R. *If these are given as the rows of a matrix* A, *suppose* $P = (a, b, c)$, $Q = (d, e, f)$, *and* $R = (g, h, i)$, *one writes* $\det(A)$, *and also,*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \textit{and } \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$



For example,



The 2×2 determinant is -2 as before. The 3×3 is

$$40 + 84 + 96 - 105 - 48 - 64 = 3.$$

(2.8) Disclaimer. For hand-calculation, the 'crossed diagonal' formula seems to be error-prone, and $P \cdot (Q \times R)$ seems to be more reliable.

(2.9) Properties of the 3×3 determinant.

- If A is a 3×3 matrix, write $\det A$ for its determinant. Then $\det A = \det(A^T)$ (determinant of transpose). Therefore the formulae below are true when P, Q, R are the rows of a 3×3 matrix, and also true when they are the columns.
- For example, $A = [P, Q, R]$ (implicitly, P, Q, R are the columns of A). Then $\det(A) = P \cdot (Q \times R)$.
- $(P \times Q) \cdot R = Q \cdot (R \times P) = P \cdot (Q \times R)$.
- Write this as $\det(P, Q, R)$. Thus $\det(P, Q, R) = \det(Q, R, P) = \det(R, P, Q)$.

- $\det(Q, P, R) = \det(R, Q, P) = \det(P, R, Q) = -\det(P, Q, R)$.
- The determinant is distributive in the sense that $\det(P, Q, \alpha R + \beta S) = \alpha \det(P, Q, R) + \beta \det(P, Q, S)$, etcetera.
- $\det(P, Q, R) = 0$ iff O, P, Q, R are coplanar. (This includes the cases $P = O, Q = O, R = O$, and P, Q, R collinear).

2.4 Uses of the cross product

(2.10) Cramer's Rule. A set of 3 linear equations in 3 unknowns

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

can be written as

$$Px + Qy + Rz = B$$

where P, Q, R, and B are column vectors. There is a form of *Cramer's Rule* valid for systems of 3 equations. The explanation is very similar to the 2×2 case, and we skip it, only stating the formulae.[2]

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

(2.11) For example, solve

$$\begin{aligned} x + 2y + 3z &= 2 \\ 4x + 5y + 6z &= 5 \\ 7x + 8y + 8z & \end{aligned}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{vmatrix} = 3 \quad \begin{vmatrix} 2 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 8 \end{vmatrix} = 3 \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 8 \end{vmatrix} = -3 \quad \begin{vmatrix} 1 & 2 & 2 \\ 4 & 5 & 5 \\ 7 & 8 & 7 \end{vmatrix} = 3$$

$$x = \frac{3}{3} = 1 \quad y = \frac{-3}{3} = -1 \quad z = \frac{3}{3} = 1$$

[2] **CORRECTED** 8/5/20. The first column in numerator for y, z was actually the first row.

2.5 Adjoint and inverse of 3×3 matrix

Let P, Q, R be the **columns**[3] of a 3×3 matrix A. The *adjoint matrix* $\text{adj}(A)$ is the 3×3 matrix whose **rows** are $Q \times R, R \times P, P \times Q$, in that order.[4]

The adjoint matrix is a multiple of the inverse matrix, and it can be used to invert a matrix. For example, invert

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad R = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix},$$

$$(Q \times R)^T = [(5)(8) - (8)(6), (8)(3) - (2)(8), (2)(6) - (5)(3)] = [-8, 8, -3]$$

$$(R \times P)^T = [10, -13, 6]$$

$$(P \times Q)^T = [-3, 6, -3]$$

$$\text{adj}(A) = \begin{bmatrix} -8 & 8 & -3 \\ 10 & -13 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

$$A \; \text{adj}(A) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

In other words, for this example $A \; \text{adj}(A) = \det(A) I_{3\times 3}$ so $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

$$A^{-1} = \begin{bmatrix} -\frac{8}{3} & \frac{8}{3} & -1 \\ \frac{10}{3} & -\frac{13}{3} & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

This is the adjoint form of the inverse:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

2.6 Adjoint, calculations organised differently

(2.12) Definition *Let $A_{n\times n}$ be a square matrix. For $1 \le i, j \le n$, the (i, j)-*minor *of A is the* determinant *of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column from A.*

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[3]The notes have been changed here; in the previous version P, Q, R were the rows. The calculations are much the same.

[4]The order is crucial; also $Q \times R$, not $R \times Q$, etcetera.

Given

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

for $1 \leq i, j \leq 3$, the (i, j)-*minor* of A is the *determinant* of the 2×2 *matrix* obtained by deleting the i-th row and j-th column from A. The minors are

$$\begin{bmatrix} \begin{vmatrix} e & h \\ f & i \end{vmatrix} & \begin{vmatrix} b & h \\ c & i \end{vmatrix} & \begin{vmatrix} b & e \\ c & f \end{vmatrix} \\ \begin{vmatrix} d & g \\ f & i \end{vmatrix} & \begin{vmatrix} a & g \\ c & i \end{vmatrix} & \begin{vmatrix} a & d \\ c & f \end{vmatrix} \\ \begin{vmatrix} d & g \\ e & h \end{vmatrix} & \begin{vmatrix} a & g \\ b & h \end{vmatrix} & \begin{vmatrix} a & d \\ b & e \end{vmatrix} \end{bmatrix}$$

Given the earlier example of A, the minors are

$$A : \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}, \quad \text{minors} \begin{bmatrix} -8 & -10 & -3 \\ -8 & -13 & -6 \\ -3 & -6 & -3 \end{bmatrix}$$

(2.13) Definition *Let $A_{n \times n}$ be a square matrix. For $1 \leq i, j \leq n$, the (i, j)-*cofactor *of A is the (i, j)-minor multiplied by $(-1)^{i+j}$.*

With the above example, the matrix of cofactors is

$$\begin{bmatrix} -8 & 10 & -3 \\ 8 & -13 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

The transpose of the matrix of cofactors is

$$\begin{bmatrix} -8 & 8 & -3 \\ 10 & -13 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

(2.14) Proposition *The adjoint of a 3×3 matrix is its matrix of cofactors,* **transposed**.
▮

2.7 Using the cross-product for the plane through 3 non-collinear points

This was done in MAU1S11.

2.8 Determinant as volume

There is a long word, parallelopiped (?) for the 3-dimensional analogue of a parallelogram. A cube is a simple example, and any parallelopiped has 6 faces like a cube, but on a cube they are square faces and on a parallelopiped they are parallelograms.

Fact: $\det(P, Q, R)$ is $\pm$(volume of parallelopiped) whose corners are $O, P, Q, R, P+Q, Q+R, R+P, P+Q+R$, and whose sign is positive if P, Q, R form a right-handed system, negative of P, Q, R form a left-handed system, and zero if O, P, Q, R are coplanar.

Volume of a tetrahedron. Let $OPQR$ be a tetrahedron whose corners are O, P, Q, R, and let V be the volume of the parallelopiped with these corners + 4 more. It can be shown that the volume of the tetrahedron is

$$\frac{V}{6}.$$