Figure 1: oblique coordinates

7 Bases and coordinate systems

7.1 In two dimensions

- The (x, y) coordinates in the Cartesian plane are the most common way of identifying point in the plane $\mathbb{R}^2$.

- Sometimes it helps to use 'oblique coordinates.' An example of oblique coordinates is given in Figure **??**.

- Roughly speaking, the cartesian coordinate system puts a square grid on the plane, and the coordinates of a point put it accurately within one of the squares. The (x, y) coordinates of a point R can be got by dropping perpendiculars onto the x- and y-axes and measuring the distance of the 'feet' of these perpendiculars from the origin, and adjusting for sign.

- The oblique coordinate system has another two coordinate axies through the origin, one red and one green. The oblique coordinates (α, β) of the same points R can be got by drawing lines parallel to each oblique axis and measuring where they cut the other obliqe axis.

  Roughly speaking, the oblique coordinate system puts a grid of parallelograms on the plane and the oblique coordinates of R put it accurately within one of the parallelograms.

- To convert one set of coordinates to another is easy.

  Let P and Q be the points whose oblique coordinates are $(1, 0)$ and $(0, 1)$, respectively. Then
  $$R = x(1, 0) + y(0, 1) = \alpha P + \beta Q$$

- Every coordinate system for $\mathbb{R}^2$ is fully defined by the pair P, Q, given in that order (otherwise (α, β) could be confused with (β, α)).

- Let S be the 2×2 matrix whose *columns* are P and Q. The calculations will be done with column vectors.

- Given that the same point R has 'old' and 'new' coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

respectively, the connection is simple

$$\begin{bmatrix} x \\ y \end{bmatrix} = S \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

- The columns P, Q can be used to define a coordinate system if and only if S is invertible. In this case we call the points P, Q (in that order) an *ordered basis* for a coordinate system and α, β the coordinates of a point relative to that basis.

7.2 In three dimensions

This is much the same, except that there would be three points P, Q, R with oblique coordinates $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively and the matrix S is 3×3.

7.3 In n dimensions

Let $U_1, \ldots, U_n$ be a list of column vectors in $\mathbb{R}^n$. Let

$$S = [U_1, \ldots, U_n]_{n \times n}]$$

be the square matrix whose columns are $U_1, \ldots, U_n$.

(7.1) Definition *If S is invertible, then $U_1, \ldots, U_n$ (in that order) is called an* ordered basis *for $\mathbb{R}^n$, and 'old' and 'new' coordinates are connected by the equations*

$$\begin{bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{bmatrix} = S \begin{bmatrix} \alpha_1 \\ \bullet \\ \bullet \\ \alpha_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha_1 \\ \bullet \\ \bullet \\ \alpha_n \end{bmatrix} = S^{-1} \begin{bmatrix} x_1 \\ \bullet \\ \bullet \\ x_n \end{bmatrix}$$

The matrix S is called the *change of basis* matrix. ***Note: some textbooks may call S^{-1} the change-of-basis matrix.***