4 Cofactor expansion along rows

Evaluating a determinant by *cofactor expansion along the* k-th row means evaluating the expression

(4.1) $$\sum_{j=1}^{n} a_{kj}(_1)^{k+j}\text{minor}_{k,j}(A)$$

Sketch Proof. Let $A = [a_{ij}]_{n\times n}$ be the matrix in question. By a series of $k-1$ swaps, the k-th row can be 'floated' up to the top of the matrix, so

```
---row 1----                ---row k----
---row 2----                ---row 1----
...              becomes    ...
...                         ...
---row k----                --- row k-1 ----
---row k+1----              --- row k+1 ---
...                         ...
```

Let A' be the modified matrix. Then

- $\det(A) = (-1)^{k-1}\det(A')$, since $k-1$ swaps were used.
- For $1 \le j \le n$, $\text{minor}_{1j}(A')$ removes the k-th row of A and the j-th column, so

$$\text{minor}_{1j}(A') = \text{minor}_{kj}(A).$$

So (remember that the 1st row of A' is the k-th of A)

$$\det(A) = (-1)^{k-1}\sum_{j=1}^{n}(-1)^{1+j}a_{kj}\text{minor}_{1j}(A')$$
$$= \sum_{j=1}^{n}(-1)^{k+j}a_{kj}\text{minor}_{kj}(A)$$

as required. ∎

Example. Evaluate

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 2 & 6 & 6 & 2 \\ 1 & 4 & 5 & 4 \\ 1 & 1 & 4 & 0 \end{vmatrix}$$

by cofactor expansion along the second row.

Answer.

$$2 \times (-1) \times \text{minor}_{2,1} + 6 \times (+1) \times \text{minor}_{2,2} + 6 \times (-1) \times \text{minor}_{2,3} + 2 \times (+1) \times \text{minor}_{2,4} =$$
$$-2 \times \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 1 & 4 & 0 \end{vmatrix} + 6 \times \begin{vmatrix} 0 & 2 & 3 \\ 1 & 5 & 4 \\ 1 & 4 & 0 \end{vmatrix} - 6 \times \begin{vmatrix} 0 & 1 & 3 \\ 1 & 4 & 4 \\ 1 & 1 & 0 \end{vmatrix} + 2 \times \begin{vmatrix} 0 & 1 & 2 \\ 1 & 4 & 5 \\ 1 & 1 & 4 \end{vmatrix} =$$
$$(-2)(25) + (6)(5) + (-6)(-5) + (2)(-5) = -50 + 60 - 10 = 0.$$

(4.2) Definition *If A is an $n \times n$ matrix, where $n \geq 2$, then for $1 \leq s, j \leq n$, the (s, j)-*cofactor *of A, $\text{cof}_{sj}A$, is*

$$\text{cof}_{sj}A = (-1)^{s+j}\text{minor}_{sj}A$$

Thus for any k,

$$\det A = \sum_{j=1}^{n} a_{kj}\text{cof}_{kj}(A)$$

Also,

(4.3) Lemma *If $1 \leq k, \ell \leq n$, and $k \neq \ell$, then*

$$\sum_{j=1}^{n} a_{kj}\text{cof}_{\ell j}(A) = 0$$

Sketch proof. Let A' be the matrix obtained from A by copying the k-th row to the ℓ-th. Then $\text{cof}_{\ell j}(A) = \text{cof}_{\ell j}(A')$, so the expression gives $\det(A')$; but A' has two equal rows and $\det(A') = 0$. ∎

(4.4) Definition *Given an $n \times n$ matrix A, the* adjoint matrix $\text{adj}(A)$ *of A is the matrix of cofactors,* **transposed***.*

(4.5) Corollary

$$A\text{adj}(A) = (\det A)I$$

and if $\det(A) \neq 0$, we get a formula for A^{-1} familiar in 2 and 3 dimensions.

(4.6) Proposition *If A is an invertible square matrix, then*

$$A^{-1} = \frac{1}{\det A}\text{adj}(A). \quad ∎$$

No proof attempted. In theory this gives a method of inverting $n \times n$ matrices for all n; in practice it is seldom useful for $n > 3$. Gauss-Jordan elimination is much more practical.

Here is one example (one needs to evaluate 100 3- and 4-factor products; this is computer-generated).

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 6 & 6 & 2 \\ 1 & 4 & 5 & 4 \\ 2 & 1 & 1 & 5 \end{bmatrix}, \quad \det A = 20,$$

$$\text{adj}A = \begin{bmatrix} 28 & 12 & -28 & 12 \\ -42 & -3 & 22 & -8 \\ 36 & 4 & -16 & 4 \\ -10 & -5 & 10 & 0 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1.4 & 0.6 & -1.4 & 0.6 \\ -2.1 & -0.15 & 1.1 & -0.4 \\ 1.8 & 0.2 & -0.8 & 0.2 \\ -0.5 & -0.25 & 0.5 & 0 \end{bmatrix}$$

Summarising some properties of the determinant.

- Zero row. If any row is zero, the determinant is zero. This follows immediately from (**??**).
- Swapping rows reverses the sign.
- If two rows are equal, the determinant is zero.
- Multilinearity: awkward to describe.

4.1 Effect of EROs (swap, scale, subtract) on the determinant

- Swap reverses sign.
- Scaling the k-th row by the constant c scales the determinant by c. This is immediate from the cofactor expansions along the k-th row.
- Subtracting from row ℓ c times row k, where $k \neq \ell$, leaves the determinant unchanged. Let A' be the matrix derived from A by this operation. Let A'' be the matrix obtained from A by replacing the ℓ-th row by the k-th. Then by cofactor expansion along the ℓ-th row,

$$det(A') = det(A) - c\det(A'')$$

where in A'' rows k and ℓ are equal, so $\det(A'') = 0$ as required. ∎