MAU11S02 seventh Monday quiz, week 9
Monday 29/3/21 due 4pm Tuesday 6/4/21

Rules and procedures.
1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard as a 'Monday assignment.'

Remember, you must show all work.

Question 1. Let

$$A = \begin{bmatrix} -5 & 2 \\ -28 & 10 \end{bmatrix}$$

(same as last week). Solve the differential equation

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix},$$

with $x = 2$ and $y = -1$ at $t = 0$.
Answer.

$$A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} =$$

$$e^{At}X_0 = \begin{bmatrix} 8e^{2t} - 7e^{3t} & -2e^{2t} + 2e^{3t} \\ 28e^{2t} - 28e^{3t} & -7e^{2t} + 8e^{3t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 18e^{2t} - 16e^{3t} \\ 63e^{2t} - 64e^{3t} \end{bmatrix}$$

Question 2. Check that your solution $X = [x(t), y(t)]^T$ to the differential equation satisfies the initial conditions (evaluate $X(0)$), and check that it satisfies the differential equation by evaluating dX/dt and AX.
Answer.

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 18 - 16 \\ 63 - 64 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{correct}$$

$$\begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} 36e^{2t} - 48e^{3t} \\ 126e^{2t} - 192e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} -5 & 2 \\ -28 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -90e^{2t} + 80e^{3t} + 126e^{2t} - 128e^{3t} \\ -504e^{2t} + 448e^{3t} + 630e^{2t} - 640e^{3t} \end{bmatrix} = \begin{bmatrix} 36e^{2t} - 48e^{3t} \\ \\ 126e^{2t} - 192e^{3t} \end{bmatrix}$$

correct.

Question 3. X and Y are independent random variables following the distribution $B(3, 1/2)$. Calculate the distribution of the pairs (X, Y) $(0 \leq X, Y \leq 3)$.
Answer.

	$P(Y)$	1/8	3/8	3/8	1/8
	Y	0	1	2	3
$P(X)$	X				
$\frac{1}{8}$	0	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$
$\frac{3}{8}$	1	$\frac{3}{64}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{3}{64}$
$\frac{3}{8}$	2	$\frac{3}{64}$	$\frac{9}{64}$	$\frac{9}{64}$	$\frac{3}{64}$
$\frac{1}{8}$	3	$\frac{1}{64}$	$\frac{3}{64}$	$\frac{3}{64}$	$\frac{1}{64}$

Question 4. X and Y are independent random variables following the distribution $B(3, 1/2)$. Calculate the distribution of the random variable $X + Y$.
Answer. Take the totals along the upper diagonals, from 0 to 6.

$X+Y$	0	1	2	3	4	5	6
P	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

Question 5. Three (independent) trials are made out of a Binomial distribution $B(3, 2/3)$ which has two outcomes, R (red, probability 2/3) and G (green, probability 1/3).

Definition: Two events X, Y are *independent* if $P(X \cap Y) = P(X)P(Y)$.

Let A be the event 'two greens,' B 'first outcome is red,' and C 'Second is green.' Calculate the probabilities of $A, B, C, A \cap B, A \cap C$, and $B \cap C$. Determine whether A and B are independent, whether A and C are independent, and whether B and C are independent. (It helps to list the 8 outcomes and label them with their probability, and which of the events A, B, C they belong to.)
Answer.

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   RRR       RRG       RGR       RGG
   8/27      4/27      4/27      2/27
   B         B         BC        ABC

   GRR       GRG       GGR       GGG
   4/27      2/27      2/27      1/27
             A         AC        C
Probabilities
   A         B         C         A and B       A and C       B and C
   2/9       2/3       1/3       2/27          4/27          2/9
Independent?                     No            No            Yes
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