MAU11S02 sixth Friday quiz ANSWERS
Friday 26/3/21 due 1pm Friday 2/4/21

Rules and procedures.

1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard as a 'Monday assignment.'

Remember, you must show all work.

Question 1. Calculate the least-squares linear estimate $y = mx + c$ for the data

$$(-3, 0),\ (-1, 1),\ (0, 2),\ (1, 2)$$

Answer.

$$X = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 11 & -3 \\ -3 & 4 \end{bmatrix}, \quad A^T Y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\text{linear } y = 19x/35 + 58/35$$

Question 2. Calculate the least-squares quadratic estimate $y = ax^2 + bx + c$ for

$$(-3, 0),\ (-1, 1),\ (0, 2),\ (1, 2)$$

same data as in Question 1.

Answer.

$$P = \begin{bmatrix} 9 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix},$$

$$A^T = \begin{bmatrix} 9 & 1 & 0 & 1 \\ -3 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} A^T A = \begin{bmatrix} 83 & -27 & 11 \\ -27 & 11 & -3 \\ 11 & -3 & 4 \end{bmatrix}, A^T Y = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\text{quadratic} \quad y = -1x^2/22 + 49x/110 + 94/55$$

Question 3. Let

$$A = \begin{bmatrix} 12 & -10 \\ 15 & -13 \end{bmatrix}$$

Calculate eigenvalues and eigenvectors for A.
Answer.

$$\begin{vmatrix} \lambda - 12 & 13 \\ 15 & \lambda + 13 \end{vmatrix} = (\lambda - 12)(\lambda + 13) - 56 = (\lambda - 2)(\lambda + 6) = 0$$

$$\lambda = 2, -3$$

$$\lambda = 2, 2I - A = \begin{bmatrix} -10 & 10 \\ -15 & 15 \end{bmatrix} \quad X_1 = \begin{bmatrix} -10 \\ -10 \end{bmatrix} \div (-10) \text{ (scaling)}$$

$$\lambda = -3, 3I - A = \begin{bmatrix} -15 & 10 \\ -15 & 10 \end{bmatrix} \quad X_2 = \begin{bmatrix} -10 \\ -15 \end{bmatrix} \div (-15) \text{ (scaling)}$$

X_1 and X_2 are eigenvectors with eigenvalues 2 and -3 respectively.

Question 4. Hence express A in the form $SA'S^{-1}$ where A' is a diagonal matrix, and evaluate e^A.
Answer.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^{-3} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3e^2 - 2e^{-3} & -2e^2 + 2e^{-3} \\ 3e^2 - 3e^{-3} & -2e^2 + 3e^{-3} \end{bmatrix}$$

There are two ways of checking this answer. First, if one pretends that $e = 1$, one should get the identity matrix. Second, $\det(e^A) = e^{\text{tr}A}$ (the trace). Working out $\det(e^A)$ is not worth the effort, but one can check this on a calculator. In this example, $\det(e^A) = e^{-1}$. Numerically, we get agreement: 0.367879.

Question 5. Use eigenvector methods to calculate

$$\arctan \begin{bmatrix} 0 & 1/3 \\ 1 & 0 \end{bmatrix}$$

Note: $\tan(\pi/6) = 1/\sqrt{3}$. It should not be necessary to know a power series expansion for $\arctan(x)$, but here it is: $x - x^3/3 + x^5/5 \ldots$ converges to $\arctan(x)$ — $\tan^{-1}(x)$ — if $|x| < 1$. The calculations are rather different this time. Sorry.

Answer. $\det(\lambda I - A) = \ldots \lambda^2 - 1/3$; $\lambda = \pm(1/\sqrt{3})$.

$$\lambda = 1/\sqrt{3} : \ldots X_1 = \begin{bmatrix} 1 \\ 1/\sqrt{3} \end{bmatrix} \quad \lambda = -1/\sqrt{3} : \ldots X_2 = \begin{bmatrix} 1 \\ -1/\sqrt{3} \end{bmatrix}$$

$$SA'S^{-1} = \begin{bmatrix} 1 & 1 \\ \sqrt{3} & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{bmatrix}$$

$$\arctan(A) = S \arctan A' S^{-1} = \begin{bmatrix} 1 & 1 \\ \sqrt{3} & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \pi/6 & 0 \\ 0 & -\pi/6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{3}} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & \frac{\pi}{6\sqrt{3}} \\ \frac{\pi\sqrt{3}}{6} & 0 \end{bmatrix}$$