Rules and procedures: this week is different.

**(1)** (See quiz 6). Solve in full the following system of differential equations.

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 17 & -12 \\ 24 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $x = 1$ and $y = 1$ at $t = 0$.

Answer.

$$\begin{bmatrix} 9e^t - 8e^{-t} & -6e^t + 6e^{-t} \\ 12e^t - 12e^{-t} & -8e^t + 9e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^t - 2e^{-t} \\ 4e^t - 3e^{-t} \end{bmatrix}$$

(2) Here is an inexpensive dart-board. One player is skilled enough to hit the board but only at random. There are four outcomes depending on which region the dart hits. The square is 6×6 and the circles have radius 3,2,1, respectively. Calculate the probability of each outcome (proportional to area).
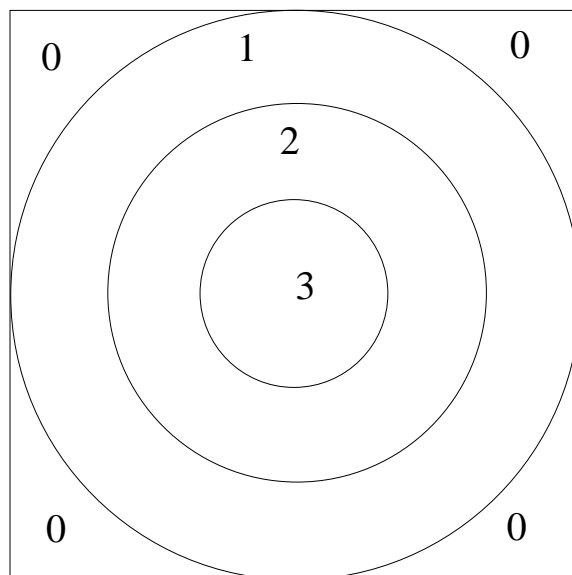
Answer. 0: $1 - \pi/4$, 1: $5\pi/36$, 2: $3\pi/36$, 3: $\pi/36$.

(3) Given two fair dice, one red, one green, suppose X is the random variable giving the number on top of the red die after throwing, Y that on the green. Let $Z = X + Y$; it takes values from 2 to 12. A be the event: Z is odd; B: Z is divisible by 3; C the event: $X < Y$.

(i) Calculate the conditional probabilities of $A|B$, $B|C$, and $C|A$ respectively. (ii) which of the following pairs of events, if any, are independent? A, B; B, C; C, A?

Answer. $P(A) = 1/2$, $P(B) = 1/3$, $P(C) = 5/12$. $P(A|B) = 1/2$, $P(B|C) = 1/3$, $P(C|A) = 5/6$. A, B are independent, B, C are not, C, A are not.

(4) The sample space consists of groups of 4 bernoulli trials whose outcomes are labelled P and Q and which have probability 3/4 and 1/4 respectively. Let A, B, C be the following events

A: Two P and two Q. B: the first outcome out of 4 is P. C: the first outcome is P and the second is Q.

(i) Calculate the conditional probabilities of $A|B$, $B|C$, and $C|A$ respectively. (ii) which of the following pairs of events, if any, are independent? A, B; B, C; C, A?

Answer. $P(A) = 27/128$, $P(B) = 3/4$, $P(C) = 3/16$. $P(A|B) = 1/6$, $P(B|C) = 1$, $P(C|A) = 1/3$. No pair is independent.

(5) Suppose that S is the sample space for some distribution. Note that S itself is an event. One of the following is true. (i) For every event A, S and A are independent. (ii) For no event A are S and A independent. Which is true?

Answer. $S \cap A = A$ and $P(S) = 1$. Then $P(S \cap A) = P(A) = P(S)P(A)$ so they are independent. (i) is true.