UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Science **Trinity Term 2017**

Mathematics 1S12, half of paper

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Attempt 3 questions from each section

1. (a) Calculate the determinant

$$\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 2 & 0 \\ -3 & -7 & -2 & -4 \\ -1 & -1 & 0 & 0 \end{vmatrix}$$

by bringing to upper triangular form.

Answer

```
 1 -1  2  3
 2 -2  2  0
-3 -7 -2 -4
-1 -1  0  0

  1   -1    2    3
  0    0   -2   -6
  0  -10    4    5
  0   -2    2    3

  1   -1    2    3
  0  -10    4    5
  0    0   -2   -6
  0    0  6/5    2

   1     -1      2      3
```

```
0  -10    4    5
0    0   -2   -6
0    0    0 -8/5
```

```
one swap determinant is 32
```

(b) Calculate the same determinant by cofactor expansion along the fourth row.

Answer

```
(-3) *(-16) + 0  + (4) * (-4) + 0 = 32
```

(c) Find bases for the row space, the column space, and the nullspace, of the following matrix:

$$\begin{bmatrix} 1 & 5 & -1 & -2 \\ -1 & -5 & 3 & 10 \\ -1 & -5 & 1 & 2 \end{bmatrix}$$

Answer

```
 1  5 -1 -2   =R1
-1 -5  3 10   +1*R1
-1 -5  1  2   +1*R1

 1  5 -1 -2   +1*R2
 0  0  2  8  *(1/2)  =R2
 0  0  0  0

 1  5  0  2
 0  0  1  4
 0  0  0  0  in rref
```

Row space basis: $[1, 5, 0, 2]$, $[0, 0, 1, 4]$. Column space basis: columns 1 and 3. $[1, -1, -1]^T$, $[-1, 3, 1]^T$.

For the nullspace, relabel x_2 as s and x_4 as t.

$$x_1 + 5s + 2t = 0$$
$$x_3 + 4t = 0$$
$$[-5s - 2t, s, -4t, t]^T : \quad \text{basis}$$
$$[-5, 1, 0, 0]^T, [-2, 0, -4, 1]^T.$$

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2. (a) Find an orthonormal basis X_1, X_2, X_3 where X_3 is on the axis $(1, -1, -1)$.
   (b) Give the matrix for rotating points 60° anticlockwise around the z-axis.
   (c) Hence or otherwise compute the matrix for rotating points 60° around the axis $(1, -1, -1)$.

Answer

$$S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A' = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$SA' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$SA'S^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

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3. Given points

$$(-1, 1),\ (0, 0),\ (1, 0),\ (3, -1)$$

   (a) Calculate the minimum least-squares linear estimate $y = mx + c$ for this data.
   (b) Calculate the minimum least-squares quadratic estimate $y = ax^2 + bx + c$ for this data.

Answer

```
det A^T A is 440
(-1, 1)   (0, 0)   (1, 0)   (3, -1)
m = -16/35, c = 12/35
a = 1/22, b = -61/110, c = 16/55
A^T A, A^T Y are
   18    8    6,    3
    8    6    2,   -1
    6    2    4,    4
```

4. (a) Answer *either* (i) *or* (ii).

   i. (*Either*): Cars arrive at a traffic lights at the rate of 4/minute. The red period is 20 seconds long. The green period is 40 seconds long, enough for up to 5 cars to get through. Given that 3 cars are left waiting when the lights turn red at time t_0,

   A. What is the probability that there will be 1 car waiting the next time they turn red, one minute after t_0?

   B. What is the probability that there will be no car waiting the next time they turn red, one minute after t_0?

   Answer

   (1) 3 arrivals.

$$e^{-4}\frac{1}{6}4^3 = 0.195367$$

   (2) ≤ 2 arrivals.

$$e^{-4}(1 + 4 + 8) = 0.238103$$

   ii. (*or alternatively*): It is thought that a coin is in some way biased, in that the probability of heads $\neq 1/2$. To test this, it is tossed 12 times.

   A. State the null hypothesis, and give a 2-tailed rejection region, a subset of $\{0 \ldots 12\}$, which achieves 5% significance.

   B. Suppose heads comes up 3 times. Do you conclude that the coin is biased, at the 5% level of significance?

Answer

```
binomial 12, .5

0 0.0002 1 0.0029 2 0.0161 3 0.0537
4 0.1208 5 0.1934 6 0.2256 7 0.1934
8 0.1208 9 0.0537 10 0.0161 11 0.0029 12 0.0002

(A) Null hypothesis: Probability is 1/2.

Rejection region: 0, 1, 2, 10, 11, 12 (total prob < 5%
                under the null hypothesis).

(B) The null hypothesis stands.
```

(b) i. Find the sample mean and sample standard deviation for
`2.13 2.82 1.24 1.15 1.49 2.25 -1.79 2.46 -1.10`

Answer

```
Sample mean 1.1833 sample variance 2.5619 sdev 1.6006
```

ii. The underlying distribution from which the above sample was drawn is $N(\mu, \sigma^2)$ where μ and σ are unknown. Using Student's t-distribution, give a 90% symmetric confidence interval for μ.

Answer

$$\left|\sqrt{9}\frac{1.1833-\mu}{1.6006}\right| \le \alpha$$
$$|1.1833-\mu| \le \alpha\frac{1.6006}{3.0000}$$
$$\mu \in \left[1.1833 \mp \frac{1.6006}{3}\alpha\right]$$
$$\alpha = 1.860 \quad (t_8)$$
$$.1910 \le \mu \le 2.1756$$

iii. Using the chi-squared distribution, give a 90% 2-tailed confidence interval for σ.

Answer

The sample variance V is 2.5619. $8S^2/\sigma^2 \sim \chi^2_8$.

Percentage points: $5\%, 2.733$; $95\%, 15.507$.

$2.733 \leq 8 \times 2.5619/\sigma^2 \leq 15.507$ with 90% confidence.

Answer

$$\left[\frac{8 \times 2.5619}{15.507}, \frac{8 \times 2.5619}{2.733}\right]$$

$$[1.3216 \leq \sigma^2 \leq 7.4991 \quad (90\%) \text{ confidence}$$

$$1.1495 \leq \sigma \leq 2.7381$$