UNIVERSITY OF DUBLIN

XMA1S121???

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Science **Trinity Term 2015**

MATHS 1S12

May . . . . . . 3 hours

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Attempt 6 questions: 3 from each section.

The official handbook of tables and formulae is available from the invigilators.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

SECTION I

1. (a) Calculate the determinant

$$\begin{vmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & -2 & -4 \\ -3 & -2 & 4 & 3 \\ 1 & 4 & -7 & -11 \end{vmatrix}$$

by bringing to upper triangular form.

Answer

```
 1   1  -2  -2  = R1
 0   1  -2  -4
-3  -2   4   3  + 3 R1
 1   4  -7 -11  - R1

1  1 -2 -2
0  1 -2 -4 = R2
0  1 -2 -3 - R2
0  3 -5 -9 - 3 R2
```

```
1  1 -2 -2
0  1 -2 -4
0  0  0  1 swap
0  0  1  3 swap

1  1 -2 -2  det = 1
0  1 -2 -4  1 swap
0  0  1  3  orig det is -1
0  0  0  1
```

(b) Calculate the same determinant by cofactor expansion along the *second* row.
Answer

```
0 0 minor n/a
0 1 minor 3
0 2 minor 0
0 3 minor 1
1(3) + 0 + (-4) 1 = -1
```

(c) By constructing the adjoint matrix — no other method — invert

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 2 & 9 & 5 \end{bmatrix}$$

Answer

```
inverse is
-1  3/2 -1/2
-2  1/2  1/2
 4 -3/2 -1/2

determinant 2
adjoint
  -2   3  -1
  -4   1   1
   8  -3  -1
```

2. (a) Let $W = (-1, 1, 1)$. Construct an orthonormal basis X_1, X_2, X_3 where X_3 is W normalised.

   (b) Give the matrix for vertical projection onto the xy-plane.

   (c) Hence, *or otherwise*, calculate the matrix for perpendicular projection onto the plane $-x + y + z = 0$.

Answer

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = SA'S^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

An easier way is

$$v \mapsto v - \frac{1}{3}(-1, 1, 1) \cdot v.$$

3. Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

   (a) Calculate eigenvalues and eigenvectors for the matrix

   (b) Calculate e^A.

   (c) Solve

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix},$$

   where $x = 5$ and $y = 5$ at $t = 0$.

Answer

$$
\begin{aligned}
(\lambda-1)(\lambda-4)+2&=0\\
\lambda^2-5\lambda+6&=0\\
\lambda&=2,3\\
\lambda=2:\lambda I-A&=\begin{bmatrix}1&1\\-2&-2\end{bmatrix},\quad\text{eigenvector}\begin{bmatrix}1\\-1\end{bmatrix}\\
\lambda=3:\lambda I-A&=\begin{bmatrix}2&1\\-2&-1\end{bmatrix},\quad\text{eigenvector}\begin{bmatrix}-1\\2\end{bmatrix}\\
S=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}&\quad S^{-1}=\begin{bmatrix}2&1\\1&1\end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}e^2&0\\0&e^3\end{bmatrix}\begin{bmatrix}2&1\\1&1\end{bmatrix}=
\begin{bmatrix}2e^2-e^3&e^2-e^3\\-2e^2+2e^3&-e^2+2e^3\end{bmatrix}
$$

$$
\begin{aligned}
\begin{bmatrix}\alpha_0\\\beta_0\end{bmatrix}=S^{-1}\begin{bmatrix}5\\5\end{bmatrix}&=\begin{bmatrix}15\\10\end{bmatrix}\\
\begin{bmatrix}\alpha\\\beta\end{bmatrix}&=\begin{bmatrix}15e^{2t}\\10e^{3t}\end{bmatrix}\\
\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&-1\\-1&2\end{bmatrix}\begin{bmatrix}15e^{2t}\\10e^{3t}\end{bmatrix}&=
\begin{bmatrix}15e^{2t}-10e^{3t}\\-15e^{2t}+20e^{3t}\end{bmatrix}.
\end{aligned}
$$

One can check that this satisfies both the equations and the initial conditions.

4. (a) The same product is manufactured on three production lines; let A, B, C be the event that an item comes from the first, second, or third line, and let D be the event that an item is defective.

Prod. line	A	B	C
Relative prod. rate	3	2	1
Cond Prob($D\|\ldots$)	.10	.09	.12

Calculate Prob($A|D$), the probability that a defective item was produced on the first line.

(b) Calculate the sample mean and sample standard deviation of the following numbers

```
-0.49 0.02 4.90 -0.13 -0.80
```

(c) Assume these numbers are normally distributed with mean μ and standard deviation σ (but μ and σ are unknown). Using Student's t-distribution, find a 90% symmetric confidence interval for the mean μ.

(d) Using the χ^2 distribution, find a 90% 2-sided confidence interval for the variance σ^2.

Answer

(a)

Event	A	B	C
Prob(E)	1/2	1/3	1/6
Prob($D\|E$)	.10	.09	.12
Prob(D)Prob(E)	.05	.03	.02

Therefore $P(D) = 0.1$ and $P(A|D) = 1/2$.

(b)

```
-0.49 0.02 4.90 -0.13 -0.80
Sample mean 0.7000 sample sdev 2.3695
```

(c) n is 5.

$$t = \sqrt{5}\frac{\overline{X} - \mu}{S}$$

t is t_4, Student's 4 degrees of freedom. Let α be its 5% point. With 90% probability, $-\alpha \leq t \leq \alpha$.

$$\alpha = 2.132$$

$$|\overline{X} - \mu| \leq \alpha \frac{S}{\sqrt{5}}$$

$$\mu \in [0.7 \mp \alpha\frac{2.3695}{\sqrt{5}} = [0.7 \mp 2.132 * 2.3695/2.236065] = [-1.559225, 2.959225]$$

with 90% confidence.

(d) **(CORRECTED)**

$$
\begin{aligned}
(n-1)\frac{S^2}{\sigma^2} &\sim \chi^2_{n-1} \\
4\frac{S^2}{\sigma^2} &= 22.45812 \\
4\frac{S^2}{\sigma^2} &\sim \chi^2_4 \\
0.7107 \leq \frac{22.45812}{\sigma^2} &\leq 9.4877 \quad 90\% \text{ probability} \\
\frac{22.45812}{9.4877} \leq \sigma^2 &\leq \frac{22.45812}{0.7107} \quad 90\% \text{ confidence} \\
\sigma^2 \in [2.367077, 31.600000] & \qquad 90\% \text{ confidence}
\end{aligned}
$$

If a 90% confidence interval is wanted for the standard deviation, just take square roots of the endpoints:

$$\sigma \in [1.538529, 5.621384] \qquad 90\% \text{ confidence}$$

SECTION II

5.