Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JF Mathematics **Trinity Term 2022**

Mathematics U11602: Computability and Logic

day **place** **time**

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Instructions to Candidates:

Attempt 3 questions
Please fold and glue the bottom right-hand corner of each answer book, to ensure anonymity.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) **(6 points).** Consider the following (unsatisfiable) set of clauses.

$$UVX,\ U\overline{V}\,\overline{X},\ \overline{U}V\overline{X},\ \overline{U}\,\overline{V}X,\ V,\ UW,\ \overline{U}\,\overline{W},\ W\overline{X},\ \overline{W}X$$

For each of the following truth-assignments, give one of the above clauses which is false under the assignment.

U	V	W	X
1	0	1	1
1	1	1	0
0	0	0	1

(b) **(8 points).** Prove by resolution that the set of clauses in 1(a) is inconsistent.

(c) **(6 points).** Produce a Turing machine which, on input x, constructs a bitstring y whose face-value $v(y)$ is the length-lex value of x. To construct y, subtract 1 from the bitstring $1x$ (1 'prepended' to x).

2. (a) **(5 points).** Consider the axioms of Peano Arithmetic with the following interpretation I: the domain is $[0, \infty)$ (nonnegative real numbers), the successor function s is interpreted as the function $s^I : \ x \mapsto x + 1$, and zero, addition, multiplication, and equality are interpreted as zero, addition, multiplication, and equality of nonnegative real numbers.

The interpretation I is ***not*** a model of PA. Identify an axiom of PA which is not true in I.

(b) **(15 points).** Show that the following formulae are theorems of PA. You may assume that addition is commutative and use equality reasoning freely. But your proofs must be otherwise formal and properly annotated.

Prove in Peano Arithmetic

i. $u \neq v \implies s(u) \neq s(v)$

ii. $s(z) + s(z) = s(s(z + z))$

iii. $s(z) + s(z) \neq s(0)$.

iv. $y \neq 0 \implies y + y \neq s(0)$.

v. Is the formula $y + y \neq s(0)$ true in the interpretation I of 2(a)? Give reasons.

3. (a) **(4 points).** Given that '$I, \sigma \models A$' has been defined, define what it means for a formula A to be (i) true, or (ii) false, in an interpretation I.

   (b) **(6 points).** Show that a formula is not necessarily true or false in I.

   (c) **(3 points).** Define a *closed* formula.

   (d) **(7 points).** Let I be an interpretation and A a *closed* formula. Show that A is either true or false in I. Cite any lemmas, etcetera, you use from the notes.

4. (a) **(3 points).** Define *recursive inseparability* of two subsets of $\mathbb{N}$.

   (b) **(4 points).** Let A and B be recursively inseparable subsets of $\mathbb{N}$, where $A \cap B = \emptyset$. Prove that A is not recursive.

   (c) **(4 points).** What if $A \cap B \neq \emptyset$?

   (d) **(4 points).** The following Turing machine, under the length-lex encoding, computes a certain function $f : \mathbb{N} \to \mathbb{N}$. What is this function?

$$q_0 0 B R q_0 \quad q_0 1 B R q_0 \quad q_0 B B R q_1$$

   (e) **(5 points).** With the same function f as in 4(d), use the Fixed Point Theorem to prove that the set A,

$$A = \{n : \ \phi_n = f\}$$

   is a nonrecursive set.