MAU11602 fifth quiz ANSWERS
Thu 15/04/21 due 11am Friday 23/4/21

Rules and procedures.
1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard.

Question 1. A property P of partial recursive functions $\phi_m()$ is any subset of the set of all partial recursive functions ϕ_m. P is *nontrivial* if P and its complement are nonempty.

Implicitly, if $\phi_m \in P$ and $\phi_{m'} = \phi_m$, then $\phi_{m'} \in P$.

Prove (Rice's Theorem) that if P is nontrivial then P is not recursive, meaning

$$\{m : \phi_m \in P\}$$

is a nonrecursive subset of $\mathbb{N}$.

Answer. If P is nontrivial then there exist a, b with $\phi_a \in P$ and $\phi_b \notin P$. Define a function f:

$$f(n) = \begin{cases} b & \text{if } \phi_n \in P \\ a & \text{if } \phi_n \notin P \end{cases}$$

By the Fixed Point Theorem, if f is recursive then there exists an n such that $\phi_{f(n)} = \phi_n$.

If $f(n) = b$, then $\phi_n \in P$, so $\phi_b \in P$ since $\phi_n = \phi_b$.

If $f(n) = a$, then $\phi_n \notin P$, so $\phi_a \notin P$ since $\phi_n = \phi_b$. Contradiction. ∎

Question 2. Is the set $\{m : \ \phi_m \quad \text{is recursive}\}$ recursive? Give reasons.

Answer. No, because the property 'ϕ_m is recursive' is a nontrivial property of partial recursive functions ϕ_m. ∎

Question 3. Show that the sets $A = \{m : \ \phi_m(n) \downarrow 0 \text{ for all } n\}$, and $B = \{m : \ \phi_m(n) \downarrow 1 \text{ for all } n\}$ are recursively inseparable.

Answer. Choose $a \in A$ and $b \in B$, and suppose that Z is a set such that $A \subset Z \subset \mathbb{N} \backslash B$. Define $f : \mathbb{N} \mapsto \{a, b\}$; $n \mapsto b$ if $n \in Z$ and $n \mapsto a$ if $n \notin Z$. By arguments similar to those in the notes, f, and therefore Z, is not recursive. ∎

Question 4. Let X and Y be recursively inseparable sets, where one of them is recursive. Prove that $X \cap Y \neq \emptyset$.

Answer. Without loss of generality, X is recursive. If $X \cap Y = \emptyset$ then $X \subseteq X \subseteq \mathbb{N} \backslash Y$ and X, Y would not be recursively inseparable. ∎

Question 5. The set of theorems of PA is recursively enumerable (given a suitable encoding of the formulae of PA as bitstrings or numbers). Prove that the set of formulae of PA which are *not* theorems of PA is not even recursively enumerable.

(Remember that if $X \subset \mathbb{N}$ is recursively enumerable and $\mathbb{N} \setminus X$ is also recursively enumerable, then X is recursive.)

Answer. Otherwise, the set of bitstrings (or numbers, length-lex) which do not encode formulae which are theorems of PA would be recursively enumerable, and the set of theorems would be recursive. ■