MAU11602 third quiz ANSWERS
Thu 11/03/21 due 11am Friday 26/03/21

Rules and procedures.
1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard.

Question 1. Granted $\exists x_i A(x_i)$ means $\neg(\forall x_i(\neg A(x_i)))$, give a simple definition of $I, \sigma \models \exists x_i A(x_i)$.
Answer. $I, \sigma \models \exists x_i A(x_i)$ if and only if there exists a domain element d such that $I, \sigma_{i \mapsto d} \models A$.

Question 2. In a theory G of groups, take f_1 to mean multiplication, f_2 means inverse, a_1 identity, and P_1 equality.

Give closed formulae using the strict language of G which express: (i) multiplication is associative, (ii) a_1 is left identity, (iii) a_1 is a right identity, (iv) f_2 is a left inverse, and (v) f_2 is a right inverse.
Answer.

$$\text{(i) } (\forall x_1(\forall x_2(\forall x_3 P_1(f_1(x_1, f_2(x_2, x_3)), f_1(f(x_1, x_2), x_3)))))$$
$$\text{(ii) } (\forall x_1 P_1(f_1(a_1, x_1), x_1))$$
$$\text{(iii) } (\forall x_1 P_1(f_1(x_1, a_1), x_1))$$
$$\text{(iv) } (\forall x_1 P_1(f_1(f_2(x_1), x_1), a1)$$
$$\text{(v) } (\forall x_1 P_1(f_1(x_1, f_2(x_1)), a1))$$

Question 3. Give an example where $A(t) \implies \exists x_i A(x_i)$ is false. Hint: $(\forall x_1 \exists x_2(x_1 \neq x_2) \implies \exists x_2(x_2 \neq x_2))$.
Answer. Apply contrapositive to the hint.

$$\begin{aligned} \neg\exists x_2(x_2 \neq x_2) &\implies \neg\forall x_2 \exists x_2(x_1 \neq x_2) \\ (\forall x_2(x_2 = x_2)) &\implies \neg(\forall x_1 \exists x_2(x_1 \neq x_2)) \\ (\forall x_2(x_2 = x_2)) &\implies (\exists x_1 \forall x_2(x_1 = x_2)) \end{aligned}$$

Take an interpretation I with domain $0, 1$ and equality interpreted as usual. Then $I \models \forall x_2(x_2 = x_2)$. On the other hand, there does not exist a domain element equal to all domain elements. The implication is false in I.

Question 4. With the above formal Group Theory, we construct an interpretation M whose domain is the set of 2×2 matrices with integer entries and determinant ± 1; f_1^M is matrix multiplication, f_2^M is matrix inverse, and a_1^M is the 2×2 identity matrix I.

Let

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 11 & -20 \\ 6 & -11 \end{bmatrix}.$$

Evaluate the truth

(i) Of the formula $P_1(f_1(x_1, x_2), x_3)$ under the snapshot $A, A, A, \ldots$

(ii) Of the same formula under snapshot $A, B, I \ldots$

(iii) Of the same formula under snapshot $A, A, I \ldots$

(iv) Of the same formula under snapshot $(C, C, I, \ldots$

Answer.

$$\begin{array}{lll} \text{(i)} & A^2 = A? & \text{false} \\ \text{(ii)} & AB = I? & \text{true} \\ \text{(iii)} & A^2 = I? & \text{false} \\ \text{(iv)} & C^2 = I? & \text{true} \end{array}$$

Note: there were two mistakes in this question. The domain was enlarged to make C, whose determinant is -1, correct. The matrix B was corrected by changing a 5 to 2, making its determinant 1.

Question 5. (i) Correct or incorrect? for any interpretation I of a theory K, and any formula A of K, either $I \models A$ or $I \models \neg A$. (ii) Correct or incorrect? for any interpretation I of a theory K, and any *closed* formula A of K, either $I \models A$ or $I \models \neg A$. Give reasons for your answers.

Answer. (i) Take domain $\mathbb{N}$ and the usual meaning for equality. The formula $x_1 = 0$ is true for some snapshots and false for others: neither A nor $\neg A$ is true. Incorrect.

(ii) Correct. Take any interpretation and let σ be any snapshot. Either $I, \sigma \models A$ or $I, \sigma \models \neg A$. If $I, \sigma \models A$ then $I, \tau \models A$ for every snapshot τ, because A is closed, so $I \models A$.

Otherwise $I, \sigma \models \neg A$ and $I \models \neg A$.