8 Truth-tables and Propositional Logic

8.1 Truth tables and propositional connectives

Propositional logic is concerned with **truth-functions**, functions whose values are the two **truth-values** $0, 1$ (for **false** and **true** respectively), and whose arguments are also truth-values.

(8.1) Definition boolean variables *are variables which are restricted to truth-values. A* boolean expression, boolean formula, *or* formula *for short, is a correctly formed expression involving boolean variables and boolean connectives.*

Certain truth-functions are well-known.

$$0 \mapsto 1, \quad 1 \mapsto 0$$

is simply *negation (not)*. If X is a boolean variable then $\neg X$ is its negation. Negation can be represented in a **truth table** as follows

X	$\neg X$
0	1
1	0

$$(0,0) \mapsto 0, \quad (0,1) \mapsto 0, \quad (1,0) \mapsto 0, \quad (1,1) \mapsto 1$$

is *conjunction (and)*. If X and Y are boolean variables, $X \wedge Y$ represents their conjunction. Here is the truth table for conjunction.

X	Y	$X \wedge Y$
0	0	0
0	1	0
1	0	0
1	1	1

It can also be displayed in a table as follows.

$X \wedge Y$	0	1
0	0	0
1	0	1

Disjunction (or) is represented $X \vee Y$ and has the following table.

$X \vee Y$	0	1
0	0	1
1	1	1

(8.2) Definition *Two formulae are* equivalent *if they have the same truth-table.*

Implication (if... then) is represented $X \Rightarrow Y$ and has the following table.

$X \Rightarrow Y$	0	1
0	1	1
1	0	1

It is just a way of connecting boolean variables, and in fact $X \implies Y$ is equivalent to $(\neg X) \vee Y$ — the two expressions have the same truth-table. I believe it is called the Philonian conditional.

(8.3) The Philonian conditional is the weakest kind of 'implication' which guarantees the following:

If X is true and $(X \implies Y)$ is true then Y is true.

(8.4) The propositional connectives have various familiar properties: $\wedge$ is commutative and associative, etcetera. Importantly,

(De Morgan laws.) $\neg(X \wedge Y)$ and $(\neg X) \vee (\neg Y)$ are equivalent, and $\neg(X \vee Y)$ and $(\neg X) \wedge (\neg Y)$ are equivalent.

(8.5) Conventions about precedence of connectives. Just as with arithmetic expressions, it is convenient to drop parentheses from boolean expressions. To avoid overload, we'll not say what they are!

8.2 Truth-functions, tautologies, and contradictions

To begin with, *every* truth-function can be realised by a boolean expression using only $\wedge, \vee, \neg$.

(8.6) Definition *If X is a boolean variable, we sometimes write $\overline{X}$ to mean $\neg X$.*

Both X and $\overline{X}$ are called literals. *If $L = \overline{X}$ then we define $X = \overline{L}$ (which is obviously correct, since $\neg\neg X$ is equivalent to X.*

We write $\pm L$ to mean L or $\overline{L}$.

(8.7) Lemma *Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function. There exists a formula formed from the boolean variables $X_1, \ldots, X_n$, which is equivalent to f.*

Half proof. Suppose that in k rows of the truth-table for f the value of f is 1. The formula has the form

$$D_1 \vee D_2 \vee \ldots \vee D_k$$

where each subformula D_i is of the form

$$(\pm X_1 \wedge \ldots \pm X_n).$$

Suppose the values in the i-th such row are $x_1, \ldots, x_j$. Then X_j occurs in the formula if $x_j = 1$, and $\overline{X}_j$ occurs if $x_j = 0$.

This breaks down if $k = 0$, so f takes the constant value 0. $X_1 \wedge \overline{X}_1$ will do in this case. ∎

(8.8) Definition *Formulae of this kind:*

$$(\pm X_{i_1} \wedge \ldots \wedge \pm X_{i_{k_1}}) \vee (\pm X_{i_{k_1}} \wedge \ldots \wedge \pm X_{i_{k_2}}) \ldots$$

are said to be in disjunctive normal form *or* DNF. *They are a disjunction of conjunctions.*

Example. Construct a DNF expression for $X_1 \Rightarrow X_2$.

X_1	X_2	$X_1 \Rightarrow X_2$
0	0	1
0	1	1
1	0	0
1	1	1

$$(\overline{X}_1 \wedge \overline{X}_2) \vee (\overline{X}_1 \wedge X_2) \vee (X_1 \wedge X_2).$$

Another example.

X_1	X_2	X_3	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0

X_1	X_2	X_3	f
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

The DNF is easily formed by picking out the rows where the f-value is 1.

$$(\overline{X_1} \wedge \overline{X_2} \wedge X_3) \vee (\overline{X_1} \wedge X_2 \wedge \overline{X_3}) \vee (X_1 \wedge \overline{X_2} \wedge X_3)$$

(8.9) Definition *A formula is in* conjunctive normal form (CNF) *if it is of the form*

$$(L_1 \vee L_2 \vee \ldots \vee L_k) \wedge (L_{k+1} \vee L_{k+2} \vee \ldots L_\ell) \wedge \ldots \wedge (L_{r+1} \vee L_{r+2} \vee \ldots L_s)$$

where $L_1, \ldots, L_s$ are literals, not necessarily distinct.

(8.10) Corollary *Every truth-function can be realised by a CNF.*

Proof. Let D be a DNF realising the *negation* of $f(T_1, \ldots, T_n)$. The formula $\neg D$ is easily converted into a CNF using De Morgan's laws, and it realises f. **Q.E.D.**