13 Snapshots and substituting u free for x_i

In this section, I will be a particular interpretation of a theory K, and D will be the domain of the interpretation.

This section is devoted to proving the following result.

(13.1) Theorem *Let K be a theory, I an interpretation, x_i a variable, σ a snapshot, $A(x_i)$ a formula, and u a term. Then*

$$I, \sigma \models A(u) \text{ if and only if } I, \sigma_{i \mapsto u^\sigma} \models A(x_i).$$

***provided that** u **is free for** x_i **in** A.*

13.1 Related result about terms

Let

$$\tau = \sigma_{i \mapsto u^\sigma}.$$

(13.2) Lemma *Let K be a first-order theory, I an interpretation, σ a snapshot, t, u terms. Then writing t as $t(x_i)$ for purposes of substitution, i.e., so $t(u)$ is defined,*

$$t^\tau = (t(u))^\sigma.$$

Proof. By induction on the depth of t. Basis:

- t a constant a_j: then $t(u) = t = a_j$ and $t^\tau = (t(u))^\sigma = (a_j)^I$.
- t a variable x_j, $j \neq i$: $t(u) = t = x_j$, and

$$t^\tau = (x_j)^\tau = \tau_j = \sigma_j = (x_j)^\sigma = (t(u))^\sigma$$

- t a variable, actually x_i. Then $t(u) = u$, and

$$t^\tau = (x_i)^\tau = \tau_i = u^\sigma = (t(u))^\sigma$$

Induction: $t = f(s_1, \ldots, s_k)$, say. Then

$$t^\tau = f^I(s_1^\tau, \ldots, s_k^\tau) = f^I((s_1(u))^\sigma, \ldots, (s_k(u))^\sigma) = (f(s_1(u), \ldots, s_k(u)))^\sigma = (t(u))^\sigma. \quad \blacksquare$$

13.2 Where x_i does not occur free in ...

(13.3) Lemma *Let x_i be a variable, σ, τ, two snapshots which are identical except that possibly $\sigma_i \neq \tau_i$. Let t be a term. If x_i does not occur in t, then*

$$t^\sigma = t^\tau.$$

Proof omitted; use induction on the depth of t. $\blacksquare$

(13.4) Lemma *Suppose that x_i is a variable, σ a snapshot, τ another snapshot identical to σ except possibly $\tau_i \neq \sigma_i$, and A a formula where x_i does not occur free in A. Then*

$$I, \sigma \models A \iff I, \tau \models A.$$

Sketch proof. By induction on the depth of A. Where A is atomic, Lemma 13.3 can be used. Inductively, A can take the forms $\neg B$, $B \Rightarrow C$, $\forall x_j B$ where $x_j \neq x_i$ and x_i does not occur free in B, nor in $(\forall x_j B)$, or finally A is $(\forall x_i B)$.

Consider only the last case.

$$I, \sigma \models A \iff$$
$$\text{for every } d \in D, \quad I, \sigma_{i \mapsto d} \models B$$

But $\sigma_{i \mapsto d} = \tau_{i \mapsto d}$.

$$\text{for every } d \in D, \quad I, \tau_{i \mapsto d} \models B \iff I, \tau \models A. \qquad \blacksquare$$

13.3 Proof of Theorem 13.1

Now for the important result.

(13.5) Lemma *Given K, I, σ, u as above, let $A(x_i)$ be a formula in which u is free for x_i. Then*

$$I, \sigma_{x_i \mapsto u^\sigma} \models A$$

if and only if

$$I, \sigma \models A(u)$$

Proof. By induction on the depth of A.

Let $\tau = \sigma_{i \mapsto u^\sigma}$.

(i) Where A is an atomic formula

$$A: \; P(s_1(x_i), \ldots, s_k(x_i))$$
$$A(u): \; P(s_1(u), \ldots, s_k(u))$$
$$I, \sigma \models A(u): \; P^I((s_1(u))^\sigma, \ldots, (s_k(u))^\sigma)$$
$$P^I(s_1^\tau, \ldots, s_k^\tau) \quad \text{(Lemma 13.2), i.e.,}$$
$$I, \tau \models A$$

(ii) Where A is $\neg B$.

$$I, \sigma \models A(u) \iff \textbf{not } I, \sigma \models B(u)$$
$$\iff \textbf{not } I, \tau \models B(x_i)$$
$$\iff I, \tau \models A.$$

(iii) Where A is $B \Rightarrow C$: $A(u)$ is $B(u) \Rightarrow C(u)$.

$$\begin{gathered}\textbf{not } I, \sigma \models (B \Rightarrow C)(u) \iff \\ I, \sigma \models B(u) \textbf{ and not } I, \sigma \models C(u) \iff \\ I, \tau \models B \textbf{ and not } I, \tau \models C \iff \\ \textbf{not } I, \tau \models A.\end{gathered}$$

(iv) Where A is $(\forall x_i B(x_i))$. Then x_i does not occur free in A, $A(u)$ coincides with A, and τ coincides with σ except possibly $\sigma_i \neq \tau_i$, so Lemma 13.4 applies.

$$\begin{gathered}I, \sigma \models A(u) \iff \\ I, \sigma \models A(x_i) \iff \\ I, \tau \models A(x_i) \quad \text{(Lemma 13.4)}\end{gathered}$$

(v) Where A is $(\forall x_j B(x_i))$, x_j differs from x_i, and x_i has no free occurrence in A nor in B. In this case, $A(u) \equiv A$ and $B(u) \equiv B$.

$$\begin{gathered}I, \sigma \models A(u) \iff \\ I, \sigma \models A \iff \\ I, \sigma_{i \mapsto t^\sigma} \models A\end{gathered}$$

Because x_i does not occur free in A, so $A^{\sigma_{i \mapsto t^\sigma}}$ and A^σ are the same.

(vi) Where A is $(\forall x_j B(x_i))$, x_j differs from x_i, and x_i occurs free in $B(x_i)$ and in A. Since x_i occurs free in the scope $(\forall x_j \ldots)$ in A, ***and u is free for x_i in A,***

$$x_j \text{does not occur in } u.$$

Suppose $I, \sigma \models A(t)$. Then for all $d \in D$,

$$I, \tau \models B(u)$$

where $\tau = \sigma_{j \mapsto d}$. By induction on depth,

$$I, \tau' \models B$$

where $\tau' = \tau_{i \mapsto u^\tau}$.

Now, x_i does not occur in u, so $u^\tau = u^\sigma$, and

$$\tau' = \tau_{i \mapsto u^\sigma}$$

Now $\tau = \sigma_{j \mapsto d}$, and

$$\tau' = \sigma_{j \mapsto d, i \mapsto u^\sigma}$$

(stretching the notation a little), or

$$\tau' = \sigma'_{j \mapsto d}$$

so

$$I, \sigma'_{j \mapsto d} \models B$$

Since d is arbitrary,

$$I, \sigma' \models A$$

Conversely, suppose $I, \sigma' \models A$. For any $d \in d$,

$$I, \tau' \models B$$

where $\tau' = \sigma'_{j \mapsto d}$. Let $\tau = \sigma_{j \mapsto d}$. Then $\tau' = \tau_{i \mapsto u^\sigma}$.

Again, $u^\sigma = u^\tau$, and

$$\tau' = \tau_{i \mapsto u^\tau}$$

whence by induction

$$I, \tau \models B(u)$$

and for any d,

$$I, \sigma_{j \mapsto d} \models B$$

so at last

$$I, \sigma \models A. \quad \blacksquare$$