12 Semantics of first-order theories

A first-order theory provides ways of *proving* certain formulae, which we will call theorems. But how do we know that the theorems are true? In Propositional logic, the criterion was simple: only tautologies are true. They are true because every truth-assignment makes them true.

So we need some idea of truth-assignment. It is much more complex than for propositional logic.

12.1 Interpretations

(12.1) Definition *Let K be a first-order theory. An* interpretation *I of $\mathcal{L}(K)$ consists of*

- *A nonempty* domain *D*
- *For each constant a_i, a constant a_i^I in D*
- *For each function letter f, a mapping $f^I: D^k \to D$, where k is the arity of f, i.e., f is a k-ary function letter.*
- *For each predicate letter P, a mapping $P^I: D^k \to \{0,1\}$, where k is the arity of P.*

Note: an interpretation of a theory may or may not satisfy its proper axioms; at this point, whether it does or doesn't is irrelevant.

Example. Taking the axioms for group theory, one could take as domain D the group of all invertible 3×3 matrices, $1^I = I_{3\times 3}$, $=^I$ equality of matrices, multiplication, which was originally represented as f_1: f_1^I is matrix multiplication, and f_2^I is matrix inverse. This interpretation makes the axioms true in a sense defined later.

Example Again group theory: Domain $\mathbb{Z}$, $1^I = 0$, f_1^I is addition, f_2^I is negation, and $=^I$ is *congruence mod 7.*

12.2 Snapshots and terms

Suppose that we have an interpretation I of Abelian groups with constant 0, addition operation, negation, and equality. What is the value of the term $x_1 + (x_2 + x_3)$? D is the domain.

The answer is that it defines or 'induces' a mapping $D^3 \to D$. For every triple in D^3, there is a well-defined value that term has.

For example, suppose that we interpret some theory of Abelian groups in terms of integers $\mathbb{Z}$. Then the above term $x_1 + (x_2 + x_3)$ translates into a function of three arguments; if these three variables are mapped to $10, 9, 8$, respectively, then the value of that term is 27.

(12.2) Definition *Let I be an interpretation of a theory K with domain D. A* snapshot *is a countable sequence*

$$d_1, d_2, \ldots$$

of elements of D.

A snapshot is intended to be a simultaneous assignment of values to *all* the variables x_i.

It is — maybe — similar or analogous to the contents of the tape in a computation step of a Turing machine. It is a 'snapshot' of the values of all variables.

- Let $\sigma = \sigma_1, \sigma_2, \ldots$ be a snapshot.
- Textbooks would call a a snapshot a 'denumerable sequence' and leave it at that.
- The variables $x_1, \ldots$ are a recognisable sequence, and σ is used *purely* as a map taking the variable x_i to σ_i.
- Or just call it an assignment, or a binding (of the variables) or a configuration, analogous in some way to a configuration of a Turing machine.

12.3 t^σ

Let I be an interpretation, t a term, and σ a 'snapshot', the value

$$t^\sigma$$

of t under σ is an element of D defined by induction on the depth of t.

- If t is a constant a,then $t^\sigma = a^I$.
- If t is a variable x_i, then $t^\sigma = \sigma_i$.
- If t is $f(t_1, \ldots, t_k)$, then t^σ is
$$f^I(t_1^\sigma, \ldots, t_k^\sigma).$$

12.4 $I, \sigma \models A$: truth-value of a formula A under a snapshot σ.

Important notation. Let I be an interpretation with domain D, σ a snapshot, d an element of D, and i a positive integer. Then

$$\sigma_{i \mapsto d}$$

is the snapshot which has value d at i and otherwise is the same as σ. That is, $\sigma_{i \mapsto d}$ is that snapshot τ such that

$$\tau_j = \begin{cases} \sigma_j & \text{if } j \neq i \\ d & \text{if } j = i. \end{cases}$$

The value (0 or 1) of a formula A under σ is defined by induction on the depth of A.

- An *atomic* formula is one of the form $P(t_1, \ldots, t_k)$, where P is a predicate letter. If A is such an atomic formula, then
$$A^\sigma \quad \text{is} \quad P^I(t_1^\sigma, \ldots, t_k^\sigma).$$
We write $I, \sigma \models A$ when A^σ is true.

- If A is $\neg B$, then A^σ is true if and only if B^σ is false.
- If A is $B \Rightarrow C$, then A^σ is true if either B^σ is false or C^σ is true.
- This is the interesting part.

  If A is $\forall x_i B$, and d is any domain element, then $\sigma_{x_i \mapsto d}$ is the snapshot σ', where

  $$\sigma'_j = \begin{cases} \sigma_j & \text{if } j \neq i \\ d & \text{if } j = i. \end{cases}$$

  Then

  $$I, \sigma \models A$$

  if and only if for every $d \in D$,

  $$I, \sigma_{x_i \mapsto d} \models B.$$