21 Representability of primitive recursive functions

(21.1) Definition : Numerals. *The* numerals *are certain terms* $\overline{n}$ *in Peano Arithmetic which correspond to the natural numbers. That is,*

$$\overline{n} = s(s(\ldots s(0)\ldots))$$

with n *occurrences of the successor function. Note* $\overline{0} = 0$*, with the first '0' being the natural number 0 and the second being the PA constant symbol.*

Warning. This is a second usage of the $\bar{i}$ notation, not to be confused with the notation used when encoding Turing machines.

Important fact. It is consistent with Peano Arithmetic that infinite integers exist. This means that one can adjoin a constant symbol a where $a > \overline{n}$ for every n. In other words, if $P(\overline{n})$ holds for every n, $\forall x P(x)$ does not necessarily follow. For that reason the three definitions below do not mean the same thing.

Important fact. The most obvious intepretation of PA is with domain $\mathbb{N}$, where

$$\begin{aligned} 0^{\mathbb{N}} &= 0 \\ s^{\mathbb{N}}: &\quad n \mapsto n+1 \\ +^{\mathbb{N}}: &\quad (m,n) \mapsto m+n \\ \times^{\mathbb{N}}: &\quad (m,n) \mapsto mn. \end{aligned}$$

The axioms of PA are plausibly true in this interpretation, so this is a model of PA, and PA is obviously consistent, though its consistency cannot be proved using just the axioms of PA (Gödel's second incompleteness theorem).

(21.2) Definition *A function* $f : \mathbb{N}^k \to \mathbb{N}$ *is* representable in PA *by a formula* $A(x_1, \ldots, x_{k+1})$ *if for every* $n_1, \ldots, n_k$,

$$\vdash_{\text{PA}} \ (\forall x_{k+1}(A(\overline{n_1}, \ldots, \overline{n_k}, x_{k+1}) \iff x_{k+1} = \overline{f(n_1, \ldots, n_k)}))$$

It is strongly representable *in PA if and only if*

$$\vdash_{\text{PA}} \ (\forall x_1, \ldots, x_{k+1}(A(x_1, \ldots, x_{k+1}) \iff x_{k+1} = f(x_1, \ldots, x_k))$$

A relation $R(n_1, \ldots, n_k)$ *on* $\mathbb{N}^k$ *is* expressible in PA *if and only if there exists a formula* $A(x_1, \ldots, x_k)$ *of PA such that*

$$R(n_1, \ldots, n_k) \iff \ \vdash_{\text{PA}} \ A(\overline{n_1}, \ldots, \overline{n_k})$$

Theorem: For every primitive recursive function $f(\vec{m})$, the relation $n = f(\vec{m})$ is expressible in PA.

Proof SKIPPED. There is a famous trick of Gödel's to generate finite sequences of numbers through his so-called Beta function. This is used to ensure that functions introduced through primitive recursion (with representable functions) are representable. It uses the Chinese Remainder Theorem to show that any finite sequence can be generated this way, but it does not require a proof within PA of the Chinese Remainder Theorem (it is not obvious how to express it in PA). ∎