17 Renaming bound variables, and other techniques

17.1 Renaming variables, and other shortcuts

(17.1) Definition *Let x_i, x_j be variables and $A(x_i)$ a formula. We say that $A(x_i)$ and $A(x_j)$ (substituting x_j for x_i) are* similar *(Mendelson) if*

- *x_j is free for x_i in $A(x_i)$, and*
- *$A(x_i)$ contains no free occurrences of x_j.*

(17.2) Lemma *If $A(x_i)$ and $A(x_j)$ are similar then $A(x_j)$ and $A(x_i)$ are similar and*

$$(\forall x_i A(x_i)) \iff (\forall x_j A(x_j)). \quad \blacksquare$$

Here are some short-cuts, which you can use *ad. lib.* to shorten and simplify proofs.

- As a matter of taste, you can use u, v, w, x, y, z as variables; avoiding too many subscripts.
- Any instance of a tautology needs no further justification.
- The Deduction Theorem, applied with full consciousness of its restrictions.
- The Fix Rule, with its restrictions.
- $\exists$-introduction:

  $$A(t) \implies \exists x_i A(x_i)$$

  if t is free for x_i in $A(x_i)$. This is a variant of axiom IV.
- Relabelling bound variables.
- Substitution instances of axioms and other theorems. That is, if it has been established that

  $$A(x_i)$$

  is true, or an axiom, then you can write down any instance $A(t)$, providing that t is free for x_i in $A(x_i)$, mentioning Axiom IV if it seems necessary.
- *Equality reasoning.* Equality theories come next. This means we can substitute equal terms in other terms and formulae getting equal terms and equivalent formulae, we can combine two equalities into one by transitivity, we can assume $x = y$ given $y = x$, and so on.

17.2 Equality Theories

An *equality theory* is a first-order theory with a distinguished binary predicate for equality, generally expressed as $=$, for which the following can be proved

- The equality relation is an equivalence relation
- If $t(x_i)$ is a term, and u, v are terms, and $u = v$, then $t(u) = t(v)$.
- If $A(x_i)$ is a formula, and u, v are terms, (both free for x_i in A), and $u = v$, then $A(u) \iff A(v)$.

Once we have an equality theory, we can have modified forms of $\exists$:

- $\exists^* x_i A(x_i) \iff (\forall x_i(\forall x_j(A(x_i) \wedge A(x_j)) \implies i = j))$.

  There exists at most one x_i such that $A(x_i)$. The variable x_j *must be chosen* so that $A(x_i)$ and $A(x_j)$ are similar.

- $\exists! x_i A(x_i) \iff (\exists x_i A(x_i) \wedge \exists^* x_i A(x_i))$.

  There exists a unique x_i such that $A(x_i)$.