14 Soundness of Predicate Calculus

14.1 Soundness of predicate calculus

(14.1) Definition *Given an interpretation I of a theory K, and a formula A,*

$$I \models A \iff \text{for every snapshot } \sigma \; I, \sigma \models A.$$

(14.2) Definition *A* predicate calculus *is a (first-order) theory with no proper axioms.*

(14.3) Lemma *Let F be a tautology in* propositional logic *with Boolean variables $X_1, \ldots, X_n$, and A a formula of first-order logic obtained by replacing the boolean variables X_i by formulae A_i. Then A is called an* instance of a tautology *and is true in every interpretation. (Exercise.)*

(14.4) Lemma *A formula A of a theory K is a* theorem *if there exists a proof of A in K.*
Recall that a predicate calculus is a first-order theory with no proper axioms.
If A is a theorem of a predicate calculus, then it is true in every interpretation.

Proof. By induction on proof length.
Proofs of length 1 are just instances of logical axioms.
Axioms of type I-III are instances of tautologies and are always true.
Suppose $(\forall x_i A(x_i)) \Rightarrow A(t)$ is an instance of Axiom IV, so t is free for x_i in A. Let I be an interpretation.
If $I \models \forall x_i A(x_i)$, then for every $d \in D$ (the domain of I), and every snapshot σ,

$$I, \sigma_{x_i \mapsto d} \models A(x_i)$$

In particular,

$$I, \sigma_{x_i \mapsto t^\sigma} \models A(x_i)$$

and by the crucial Theorem 13.1,

$$I, \sigma \models A(t)$$

Therefore

$$I, \sigma \models (\forall x_i A(x_i)) \implies A(t).$$

This holds for every snapshot σ, so

$$I \models (\forall x_i A(x_i)) \implies A(t).$$

For an axiom of type V, suppose

$$I, \sigma \models (\forall x_i (A(x_i) \Rightarrow B(x_i)))$$

where x_i has no free occurrence in A.
Equivalently, for every $d \in D$,

$$I, \sigma_{x_i \mapsto d} \models A(x_i) \Rightarrow B(x_i).$$

Assuming this, suppose that

$$I, \sigma \models A(x_i) \qquad (*)$$

Since x_i does not occur free in $A(x_i)$, for every d,

$$I, \sigma_{x_i \mapsto d} \models A(x_i)$$

(Lemma 13.2). But also

$$I, \sigma_{x_i \mapsto d} \models A(x_i) \Rightarrow B(x_i).$$

so

$$I, \sigma_{x_i \mapsto d} \models B(x_i)$$

for all d, and

$$I, \sigma \models \forall x_i B(x_i) \qquad (**)$$

So, combining (*) with (**),

$$I, \sigma \models (\forall x_i (A \Rightarrow B)) \implies (A \Rightarrow \forall x_i B)$$

That is, every instance of Axiom V is true under every snapshot σ in every interpretation I: it is always true.

Induction, MP: Suppose that B is deduced by MP from earlier formulae $A, A \Rightarrow B$. Let I, σ be any snapshot in any intepetation. By induction,

$$I, \sigma \models A \qquad \text{and} \qquad I, \sigma \models A \Rightarrow B$$

Then by definition of '$I, \sigma \models \ldots$,'

$$I, \sigma \models B$$

so B is always true.

Induction, Gen: Suppose that B is deduced from an earlier formula $C(x_i)$ using Generalisation (on x_i). By induction, C is always true. For any I, σ,

$$I, \sigma \models C(x_i)$$

Therefore, for any I, σ,

$$I, \sigma_{x_i \mapsto d} \models C(x_i)$$

for every d in the domain of I. That is,

$$I, \sigma \models \forall x_i C(x_i)$$

for every I, σ, so B is always true. ∎