25 Integers can be infinite

In this section we prove, that, assuming PA is consistent,

PA + (there exists an infinite integer) is consistent.

Or, rather, formulated as follows...

(25.1) Theorem *Let K be the extension of Peano Arithmetic obtained by adjoining a new constant a, plus the (countably many new) axioms*

$$A_n(a) \ldots \quad a \neq \overline{n}$$

for $n = 0, 1, 2 \ldots.$

Then K is consistent.

Proof. If inconsistent, then

$$\vdash_K 0 \neq 0.$$

Fix a proof Π that $0 \neq 0$. It can use only finitely many new axioms $A_n(a)$. So there exists a k such that every new axiom occurring in the proof Π is in the list $A_0(a), \ldots, A_k(a)$, and Π is also a proof that

$$\Pi : \quad A_0(a), \ldots, A_k(a) \vdash_{\text{PA}} 0 \neq 0$$

Choose a variable y not occurring in the proof Π, and replace a by y throughout the proof, including, of course, the new axioms $A_n(a)$. As in the 'neutral constant' lemma (18.5) associated with the Completeness Theorem, we get another proof, call it Π', that $0 \neq 0$. All occurrences of y in Π' are free.

$$\Pi' : \quad A_0(y), \ldots, A_k(y) \vdash_{\text{PA}} 0 \neq 0$$

In order that the deduction theorem be valid, it is necessary that no variable occurring free in any $A_j(y)$ is generalised in any step depending on $A_j(y)$. But y is the only variable occurring in $A_j(y)$, and it is generalised nowhere, so the condition is met:

$$\vdash_{\text{PA}} (A_0(y) \wedge \ldots \wedge A_k(y)) \implies 0 \neq 0$$

Take the counterpositive. Since $\vdash_{\text{PA}} 0 = 0$, we can use MP to get

$$\vdash_{\text{PA}} (\neg A_0(y)) \vee \ldots \vee \neg A_k(y)$$

Generalise:

$$\forall y(\neg A_0(y) \vee \ldots \vee \neg A_k(y))$$

That is,

$$\forall y(y = \overline{0} \vee y = \overline{1} \vee \ldots \vee y = \overline{k})$$

The term $\overline{k+1}$ is free for y in the above formula, so using Axiom IV

$$(\overline{k+1} = \overline{0} \vee \overline{k+1} = \overline{1} \vee \ldots \vee \overline{k+1} = \overline{k})$$

which is provably false in PA. ∎