18 Completeness of first-order theories

The result proved in this section is Gödel's Completeness Theorem: every consistent theory has a model. We follow Mendelson's proof, which he credits mostly to Henkin.

18.1 Complete theories

(18.1) Definition

- *If K is a first-order theory, then $\mathcal{L}(K)$ is its language, that is, the terms and formulas constructible using the constants, function letters, and predicate letters, of K, together with the variables $x_1, \ldots$.*
- *A* closed *formula is one without free variables.*
- *A theory is K* inconsistent *if there exists a closed formula A such that A and $\neg A$ are both theorems of K.*

  Else K is consistent.
- *If K is a first-order theory and A a formula (in $\mathcal{L}(K)$, then $K + A$ is the theory obtained by adding A as an extra axiom.*

Exercise. Suppose T is an inconsistent theory. Prove that *every* formula of T is provable in T.

(18.2) Lemma *If A is a closed formula of a theory T, and $\neg A$ is not a theorem of T, then $T + A$ is consistent.*

Proof. Otherwise *every* formula of T is provable, and in particular, $\neg A$.

$$A \vdash_T \neg A$$

Since A is closed, the Deduction Theorem (involving A) is valid, and the following are theorems of T:

1. $A \Rightarrow \neg A$ (Deduction Theorem).
2. $\neg\neg A \Rightarrow \neg A$ (1, contrapositive)
3. $\neg\neg A \Rightarrow A$ (proved earlier)
4. $(\neg\neg A \Rightarrow \neg A) \implies ((\neg\neg A \Rightarrow A) \implies \neg A)$ (Axiom III)
5. $((\neg\neg A \Rightarrow A) \implies \neg A)$ 2,4, MP
6. $\neg A$ (3,5,MP).

In other words, if $T + A$ is inconsistent, then $\neg A$ is a theorem of T. ∎

(18.3) Definition *A theory is* complete *if, for every* closed *formula A, either A or $\neg A$ is a theorem.*

A consistent complete extension *of a theory T is a theory T' such that*

- $\mathcal{L}(T') \supseteq \mathcal{L}(T)$, *i.e., every term and formula of* T *is one of* T',
- *Every theorem of* T *is a theorem of* T'.
- T' *is complete and consistent.*

(18.4) Corollary *Every consistent theory* T *has a complete consistent extension with the same language as* T.

Proof. Let $A_1, \ldots$ be an enumeration of the closed formulae of T in any order. Build a sublist of these formulae as follows.

$$S_0 = \varnothing$$

$$S_{n+1} = \begin{cases} S_n & \text{if} \quad S_n \vdash_T \neg A_{n+1} \\ S_n \cup \{A_{n+1}\} & \text{otherwise} \end{cases}$$

Using an inductive argument, and the above lemma, $T + S_n$ is consistent for every n. Let T' be the extension of T obtained by adding as axioms every formula in every S_n. Using the above lemma, for each n, $T + S_n$ is consistent.

If T' were inconsistent, then, since only finitely many axioms are used to prove a contradiction, $T + S_n$ would be inconsistent for some n. Hence T' is consistent.

For every n, $S_n \vdash_T A_n$ or $S_n \vdash_T \neg A_n$. hence either A_n or $\neg A_n$ is a theorem of T'. Since T and T' have the same language, T' is complete. ∎

18.2 Adding new constants

Suppose that K is a first-order theory and a a new letter, to be used as a constant. $K + a$ is a theory obtained by (i) adding a as a new constant; $\mathcal{L}(K + a)$ is an extension of $\mathcal{L}(K)$ and (ii) adding all new instances of logical axioms I–V as required for the extended language. The *proper* axioms of $K + a$ are the same as for K.

We call a a 'neutral constant.'

(18.5) Lemma *If* K *is consistent, then* $K + a$ *is consistent.*

Proof. Otherwise everything is a theorem: choose a closed formula X, without loss of generality not mentioning a, so that $X \wedge \neg X$ is a theorem of $K + a$.

Choose a fixed proof of this formula in $K + a$.

Choose any variable y not mentioned in the proof.

Replace every occurrence of a in every step of the proof, by y.

This operation (a) replaces logical axioms by other logical axioms, (b) leaves proper axioms untouched (they don't involve a), and replaces valid applications of MP and Gen by valid applications of MP and Gen (note y is never generalised).

We get a proof of $X \wedge \neg X$ in K, so K was inconsistent. ∎

18.3 Scapegoats

(18.6) Definition *Let T be a theory. Suppose that $A(x_i)$ is a formula of T in which x_i is the only free variable. A* scapegoat *for this formula is a* closed term *(constant term, ground term, variable-free term) t such that*

$$\exists x_i \neg A(x_i) \implies \neg A(t)$$

(18.7) Lemma *Given a consistent theory T and a formula $A(x_i)$ with just one free variable x_i, one can extend the theory if necessary to produce a consistent theory T' in which that formula has a scapegoat.*

Proof. If T doesn't already have a scapegoat for the formula, adjoin a new constant letter b and extend the language accordingly. The theory T'

- Has the language extended by adjoining the new constant letter b.
- Has all logical axioms (I-V) as needed for the extended language.
- Has all the proper axioms of T.
- So far, we have $T + b$. There is the additional *proper* axiom

$$X(b) : \quad (\exists x_i \neg A(x_i)) \implies \neg A(b).$$

- $T' = T + b + X(b)$.

If the new system is inconsistent, then the negation of the new axiom $X(b)$ is provable in T'. Therefore, the negation is deducible from $X(b)$ in $T + b$. So:

$$X(b) \vdash_{T+b} \exists x_i \neg A(x_i)$$
and
$$X(b) \vdash_{T+b} A(b).$$

Take the combined proof of these results and select a variable y which is mentioned nowhere in the proof. Replacing b by y throughout we get a proof of the formula $A(y)$, deduced from $X(y)$ in T.

$$X(y) \vdash_T \exists x_i \neg A(x_i)$$
and
$$X(y) \vdash_T A(y).$$

Let U be the formula $\exists x_i \neg A(x_i)$ and V the formula $\neg A(y)$. So $X(y)$ is $U \implies V$.

$$U \implies V \vdash_T U$$

and y is the only free variable in U and is never generalised, so the Deduction Theorem is valid.

$$\vdash_T (U \implies V) \implies U$$

Now whenever U is false, the above compound formula is false, so

$$((U \implies V) \implies U) \implies U$$

is a tautology, so

$$\vdash_T U : \quad \vdash_T \exists x_i \neg A(x_i)$$

Also,

$$U \implies V \vdash_T A(y)$$
$$U \implies V \vdash_T \neg V$$
$$\vdash_T (U \implies V) \implies \neg V$$

But the last formula is false if V is true, so

$$\vdash_T \neg V$$
$$\vdash_T A(y)$$

Generalise:

$$\vdash_T \forall y A(y)$$

whereas

$$\vdash_T \exists x_i \neg A(x_i),$$

and T would be inconsistent. ∎

(18.8) Corollary *If T is a consistent theory, it can be extended to a consistent scapegoat theory.*

Proof. The process in the above lemma can be repeated (countably often) to produce a scapegoat theory as follows. Adjoint countably many new constants $b_1, \ldots$; let $F_i(y_i)$ be an enumeration of all the formulae *in the extended language* which have exactly one free variable y_i. The extension is done in stages; at the ith stage a scapegoat is, if necessary, created for $F_i(y_i)$. To do this, let $b_{j(i)}$ be the earliest of the new constants which is not mentioned in $F_1(b_{j(1)}), \ldots, F_{i-1}(b_{j(i-1)})$, and add the new proper axiom

$$(\exists y_i \neg F(y_i)) \implies \neg F_i(b_{j(i)})$$

Ultimately we get a scapegoat theory. ∎

18.4 Consistent complete scapegoat theories

(18.9) Lemma *Let T be a complete consistent scapegoat theory. Let D be the set of all closed terms (constant terms, ground terms, variable-free terms) of T. Define an interpretation M of T as follows.*

- *If a is a constant of T then $a^M = a$.*
- *If f is an n-ary function symbol (letter) of T, then*

$$f^M : \quad (c_1, \ldots, c_n) \mapsto f(c_1, \ldots, c_n)$$

  (Note that c_j are closed (variable-free) terms.)

- *If P is an n-ary predicate of T, then*

$$P^M : \quad (c_1, \ldots, c_n) \mapsto \begin{cases} 1 & \textit{if} \vdash_T P(c_1, \ldots, c_n) \\ 0 & \textit{otherwise} \end{cases}$$

Remark. *Since T is consistent and complete, and $P(c_1, \ldots, c_k)$ is closed,* **not** $\vdash_T P(c_1, \ldots, c_n)$ *if and only if $\vdash_T \neg P(c_1, \ldots, c_n)$.*

Then for any formula A of T, and snapshot σ,

$$M, \sigma \models A \quad \textit{if and only if} \quad \vdash_T A^\sigma$$

where A^σ is the closed formula obtained by subsituting σ_i for every free occurrence of x_i in A.

Proof.

Snapshots are sequences of terms. Therefore, for any expression E, and snapshot σ, one can write

$$E^\sigma$$

for the result of replacing every free occurrence of a variable x_i in E by the term σ_i. E^σ is an expression of T, whereas $E^{M,\sigma}$ could be regarded (if E is a formula) as synonymous with $M, \sigma \models E$ and be just a truth-value, 0 or 1.

We claim: for every formula A and snapshot σ,

$$M, \sigma \models A \quad \text{if and only if} \quad \vdash_T A^\sigma$$

The claim is proved by induction on the depth of A.

- If A is atomic, i.e., has the form

$$P(t_1, \ldots, t_n)$$

  then each t_j^σ is a closed term and belongs to D.

$$M, \sigma \models P(t_1, \ldots, t_n)$$

  if and only if

$$P^M(t_1^\sigma, \ldots, t_n^\sigma)$$

i.e.,

$$\vdash_T P(t_1^\sigma, \ldots, t_n^\sigma)$$

i.e.

$$\vdash_T A^\sigma$$

- A is $\neg B$.

$$M, \sigma \models A \quad \text{if and only if not } M, \sigma \models B$$

By induction,

$$M, \sigma \models B \text{ if and only if } \vdash_T B^\sigma$$

Also,

$$\vdash_T A^\sigma \text{ if and only if not } \vdash_T B^\sigma$$

('if' because T is complete and 'only if' because T is consistent). Therefore

$$M, \sigma \models A \text{ if and only if} \vdash_T A^\sigma$$

- A is $B \Rightarrow C$.

If *not* $M, \sigma \models B \Rightarrow C$, then $M, \sigma \models B$ and not $M, \sigma \models C$. By induction, $\vdash_T B^\sigma$ and not $\vdash_T C^\sigma$, and by completeness of T, $\vdash_T \neg C^\sigma$.

The following are tautologies, and therefore theorems, for any formulae X, Y.

$$X \implies ((\neg Y) \implies \neg(X \implies Y)$$
$$(\neg(X \implies Y)) \implies X$$
$$(\neg(X \implies Y)) \implies \neg Y$$

Whence

$$B^\sigma, \neg C^\sigma \vdash_T \neg(B \implies C)^\sigma$$

so $\vdash_T \neg A^\sigma$ and not $\vdash_T A^\sigma$.

If *not* $\vdash_T B^\sigma \Rightarrow C^\sigma$, $\vdash_T \neg(B^\sigma \Rightarrow C^\sigma)$, so $\vdash_T B^\sigma$ and $\vdash_T \neg C^\sigma$ from above, so *not* $\vdash_T C^\sigma$ (consistency), by induction $M, \sigma \models B$ and not $M, \sigma \models C$, so *not* $M, \sigma \models B \Rightarrow C$.

- A is $\forall x_i B$.

The variable x_i may or may not occur free in B. Write $\vec{y}$ for the *other* variables occurring free in B; possibly $\vec{y}$ is empty.

Then A^σ may be written as $\forall x_i B(\vec{y}^\sigma, x_i)$.

First suppose A^σ is a theorem. Using a type IV axiom and MP, for any $u \in D$ (u is a closed term),

$$\vdash_T B(\vec{y}^\sigma, u)$$

In this case, by induction, for every $u \in D$,

$$M, \sigma_{i \mapsto u} \models B(\vec{y}, x_i)$$

since u is arbitrary,

$$M, \sigma \models \forall x_i B(\vec{y}, x_i)$$
$$M, \sigma \models A$$

Next suppose that A^σ is not a theorem. Then since T is complete,

$$\vdash_T \neg\forall x_i B(\vec{y}^{\,\sigma}, x_i)$$

a formula equivalent to

$$\exists x_i \neg B(\vec{y}^{\,\sigma}, x_i)$$

But only x_i can be free in $B(\vec{y}^{\,\sigma}, x_i)$.

If x_i has free occurrences in this formula, then, since T is a scapegoat theory, there exists a closed term u such that

$$\vdash_T \neg B(\vec{y}^{\,\sigma}, u)$$

So *not* $\vdash_T B(\vec{y}^{\,\sigma}, u)$, and *not* $M, \sigma_{i\mapsto u} \models B$ (induction), so *not* $M, \sigma \models \forall x_i B(\vec{y}, x_i)$: *not* $M, \sigma \models A$.

If x_i does not occur free in $B(\vec{y}^{\sigma}, x_i)$, then that formula is closed, and also

$$\exists x_i \neg B(\vec{y}^{\,\sigma}, x_i) \vdash_T \neg B(\vec{y}^{\sigma}, x_i)$$

using the Fix Rule (x_i doesn't occur free in the conclusion).

$$\vdash_T \neg B^\sigma$$
$$\textit{not } \vdash_T B^\sigma$$
$$\textit{not } M, \sigma \models B$$

(induction), and again *not* $M, \sigma \models A$. ∎

(18.10) Corollary *Every consistent complete scapegoat theory T has a model.*

Proof. Without loss of generality, all proper axioms are closed. If A is a proper axiom, then $\vdash_T A$. Since it is closed, $A^\sigma = A$ for every snapshot σ, so $M, \sigma \models A$. Since σ is arbitrary, $M \models A$. ∎

18.5 The completeness theorems

(18.11) Definition *A predicate calculus is a first-order theory with no proper axioms, i.e., only logical axioms of groups I–V.*

Given a fixed first-order language, and a formula A, write

$$\models A$$

to mean that $M \models A$ for every *interpretation M. Or, A is true in* every *interpretation.*

In this case, one says that A is logically valid.

(18.12) Theorem (i) *Every consistent complete scapegoat theory has a countable (or finite) model.*
(ii) *Every consistent theory has a countable (or finite) model.*
(iii) *Let P be a predicate calculus. A formula (in its language) is a theorem of P if and only if it is logically valid.*
(iv) *If T is a (consistent) theory, and A a formula of T, then A is a theorem of T if and only if it is true in every model of T.*

Proof. (i) From Corollary 18.10. 'Countable model' means one whose domain is countable. This was not mentioned in the lemma, but it is obvious.

(ii) If K is a consistent theory, it can be extended to a consistent scapegoat theory K' which can be extended to a complete consistent theory T. The language of T and K' are the same, so T is also a scapegoat theory and has a model, which satisfies every axiom of T, and therefore of K, so it is a model of K.

(One small point: there has to be at least one constant, otherwise there are no closed terms. However, in this case a 'neutral constant' can be included without violating consistency.)

(iii) Logical axioms are true in every interpretation, so every axiom of P, and therefore every theorem of P, is true in every interpretation. Conversely, suppose A is closed (a small adjustment will get past this restriction). If A is not a theorem, then $P + \neg A$ is consistent and has a model in which A is false.

(iv) Every theorem of T is true in every model of T. Conversely, if A is a closed formula not a theorem of T then $T + \neg A$ is consistent and has a model in which A is false. ∎