16 The 'fix' rule.

The 'fix rule' is a variant of Mendelson's 'choice rule.' It supports the following kind of argument in everyday mathematics:

$$\ldots \exists x A(x) \ldots$$
$$\text{Fix } x \text{ so that } A(x)$$
$$\ldots$$

In first-order logic we need to be careful about how this rule is used.

(16.1) Theorem *Suppose*

$$\exists x_i A(x_i)$$

has been proved. We can 'Fix x_i' and assume

$$A(x_i)$$

If we deduce a formula B, and

- *The conditions of the Deduction Theorem have been met, i.e., no variable occurring free in $A(x_i)$ have been generalised in any step depending on $A(x_i)$, and*
- *x_i does not occur free in B,*

then B can be deduced from

$$\exists x_i A(x_i).$$

Proof. According to the Deduction Theorem,

$$\vdash_K A(x_i) \Rightarrow B$$

and through contrapositive,

$$(\neg B) \Rightarrow \neg A(x_i)$$

Generalise.

$$(\forall x_i (\neg B) \Rightarrow \neg A(x_i))$$

But x_i is not free in B, so using Axiom V and MP,

$$(\neg B) \Rightarrow (\forall x_i \neg A(x_i))$$

and through contrapositive

$$(\neg \forall x_i \neg A(x_i)) \Rightarrow \neg\neg B$$

That is

$$(\exists x_i A(x_i)) \Rightarrow \neg\neg B$$

and $\neg\neg B \Rightarrow B$, so

$$(\exists x_i A(x_i)) \Rightarrow B$$

and

$$\exists x_i A(x_i) \vdash_K B. \quad \blacksquare$$