# 11 First-order languages and theories

First-order theories have

- A *language,* namely, its *terms* and *formulae.*
- 5 schemes of logical axioms, including the three from SC.
- Two rules of inference.
- Proper axioms.

The 'expressions,' the language, are built from *constant letters, variables, function letters,* and *predicate letters.* Each function letter carries an 'arity,' a positive integer, the number of arguments it takes. Likewise a predicate letter.

Also, commas and parentheses, connectives $\neg$ and $\implies$, and $\forall$.

**Example.** The variables are usually written as $x_1, x_2, \ldots$ Constant $a_1$, two functions $f_1$ (binary) and $f_2$ (unary), and one predicate $P_1$ (binary).

The proper axioms are

$$\forall x_1 P_1(f_1(x_1, a_1), x_1)$$
$$\forall x_1 P_1(f_1(a_1, x_1), x_1)$$
$$\forall x_1 P_1(f_1(f_2(x_1), x_1), a_1)$$
$$\forall x_1 P_1(f_1(x_1, f_2(x_1)), a_1)$$
$$\forall x_1 \forall x_2 \forall x_3 P_1(f_1(x_1, f_1(x_2, x_3)), f_1(f_1(x_1, x_2), x_3))$$

## 11.1 First-order languages

### 11.1.1 Terms

This resemble arithmetic expressions.

- Every constant is a term on its own. It has 'depth' zero.
- Every variable is a term on its own. It has 'depth' zero.
- If $f$ is a function letter of arity $k$, and $t_1, \ldots, t_k$ are terms, then

  $$f(t_1, \ldots, t_k)$$

  (an string of letters, variables, constants, commas, and parentheses) is a term, and the depth of this term is

  $$1 + \max_{1 \leqslant j \leqslant k} \text{depth}(t_j)$$

### 11.1.2 Term substitution

Let $t$ be a term, $x_i$ a variable which may or may not occur in $t$. We write $t(x_i)$ **to guide substitution.**

Let $u$ be any term.

$$t(u)$$

is the term obtained by substituting the term $u$ for $x_i$, under the following recursive rules.

- Write $t(x_i)$ for the term $t$.

  If $t$ is a constant $a_j$, then $t(u) = a_j$, no change.
- if $t$ is a variable $x_j$ and $x_j \neq x_i$ then $t(u) = x_j$, no change.
- If $t$ is $x_i$, then $t(u) = u$.
- If $t$ is $f(s_1, \ldots, s_k)$, arity $k$ and $k$ arguments $s_j$ which are terms, then

$$t(u) = f(s_1(u), \ldots, s_k(u))$$

### 11.1.3 Formulae

- An *atomic formula* is an expression

$$P(t_1, \ldots, t_k)$$

  where $P$ is a predicate letter of arity $k$ and $t_1, \ldots, t_k$ are terms. It has depth 0.
- If $A$ is a formula, so is $(\neg A)$: its depth is $1 + \text{depth}(A)$.
- If $A$ and $B$ are formulae, so is $(A \Rightarrow B)$. Its depth is $1 + \max(\text{depth}(A), \text{depth}(B))$.
- If $A$ is a formula, so is $(\forall x_i A)$: its depth is $1 + \text{depth}(A)$.

### 11.1.4 Other connectives

$(\exists x_1 A)$ is an abbreviation for

$$(\neg(\forall x_1(\neg A))).$$

Also, $\vee, \wedge, \iff$ are introduced by definition, as in Propositional Logic.

### 11.1.5 Substitution in formulae

Let $A(x_i)$ be a formula, $u$ a term.

$$A(u)$$

is defined by induction on depth as follows.

- Atomic formula $A = P(s_1, \ldots, s_k)$; $A(\vec{t}) = P(s_1(u), \ldots, s_k(u))$.
- Negation: $(\neg A)(u) = (\neg A(u))$.

- Implication: $(A \Rightarrow B)(u) = (A(u) \Rightarrow B(u))$.
- $(\forall x_j A)(\vec{t})$: if $x_j$ is *not* $x_i$, then

$$(\forall x_j A)(u) = (\forall x_j A(u)).$$

- $(\forall x_j A)(\vec{t})$: if $x_j$ *is* $x_i$, then

$$(\forall x_i A)(u) = (\forall x_i A)$$

  (no change).

### 11.1.6 Scope of a quantifier, free and bound occurrences

**(11.1) Definition** *Suppose that a quantified formula $(\forall x_i B)$ occurs as part of a formula $A$. That part of the formula $A$ between the parentheses $(\ldots)$ within the quantified formula $(\forall x_i \ldots)$ is called the* scope *of the given quantifier $\forall x_i$.*

*An occurrence of $x_i$ is* bound *if it occurs within the scope of a quantifier $(\forall x_i \ldots)$, in which the variable $x_i$ itself is quantified. Otherwise it is* free*.*

*If there is at least one free occurrence of $x_i$ in $A$ then we say $x_i$* occurs free *in $A$.*

We can describe $A(u)$ more briefly as follows.

**(11.2) Lemma** *The formula $A(u)$ is obtained from $A(x_i)$ by replacing every* free *occurrence of $x_i$ in $A(x_i)$ by $u$.* ∎

### 11.1.7 Term $t$ free for $x_i$ in $A(x_i)$

The following definition is very important.

Given a formula $A(x_i)$ and a term $t$, it is possible that there exists a variable $x_j$ in $t$ and a free occurrence of $x_i$ in $A$ such that the occurrence of $x_i$ is within the scope of a quantifier $\forall x_j$

In that case, if we substitute $t$ for $x_i$, then an occurrence of $x_j$ becomes bound.

Otherwise $t$ is *free for $x_i$* in $A$.

**(11.3) Definition** *Again: $t$ is* free for $x_i$ in $A$ *if* no free *occurrence of $x_i$ in $A$ is within the scope $(\forall x_j \ldots)$ of a quantifier where $x_j$ occurs in $t$.*

## 11.2 First-order theories

A *first-order theory* $K$ is a system with the following features.

- A first-order language $\mathcal{L}(K)$.
- Five schemes of logical axioms.
- Two inference rules: MP and Generalisation. Generalisation (gen) is

$$A \vdash_K (\forall x_i A)$$

- Proper axioms.[1]

The logical axioms are

(I.) $A \Rightarrow (B \Rightarrow A)$

(II.) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

(III.) $((\neg A) \Rightarrow \neg B) \Rightarrow (((\neg A) \Rightarrow B) \Rightarrow A)$

(IV.) Only when $t$ is free for $x_i$ in $A(x_i)$:

$$(\forall x_i A(x_i)) \Rightarrow A(t)$$

(V.) Only when $x_i$ does not occur free in $A$:

$$(\forall x_i (A \Rightarrow B)) \Rightarrow (A \Rightarrow (\forall x_i B))$$

**Example.** Where Axiom IV does not apply:

$$(\forall x_1 (\exists x_2 (x_1 \neq x_2))) \implies (\exists x_2 (x_2 \neq x_2))$$

[1]If there are no proper axioms then the theory is called a predicate calculus.