15 Deduction Theorem for first-order logic

15.1 The Deduction Theorem

In any first-order theory T, generalisation permits the inference

$$A \vdash_T \forall x_i A$$

However, this is not to say that $\vdash_T A \Rightarrow (\forall x_i A)$. Indeed, this last formula need not be logically valid.

(15.1) One can produce examples where $A \vdash_T B$, and $A \Rightarrow B$ is false in at least one interpretation of T, and hence $A \Rightarrow B$ is not a theorem of T. In other words, the Deduction Theorem for zero-order logic does not always hold in first-order theories. There is a Deduction Theorem, and it is very useful, but it has restrictions which are not found in Sentential Calculus.

(15.2) Definition *Let P be a deduction from Γ, A in a first-order theory T. An occurrence of B in one of the proof steps* depends on A *if either $B = A$, and the justification is that it is a given formula, or B is deduced from $C, C \Rightarrow B$ using Modus Ponens, where C or $C \Rightarrow B$ depends on A, or B is deduced from C using Generalisation, where C depends on A.*

(15.3) Lemma *Suppose $\Gamma, A \vdash B$ in a proof where the considered occurrence of B does not depend on A. Then $\Gamma \vdash_T B$.*

Proof. By induction on the length of proof. If $B \in \Gamma$ or B is an axiom of T then $\Gamma \vdash B$. If B is deduced from two earlier formulae $C, C \Rightarrow B$ not depending on A, using Modus Ponens, then by induction $\Gamma \vdash_T C$ and $\Gamma \vdash C \Rightarrow B$, so $\Gamma \vdash B$ using Modus Ponens. If B is deduced from an earlier formula C using Generalisation, where C does not depend on A, then $\Gamma \vdash C$ by induction, so $\Gamma \vdash B$ by Generalisation. **Q.E.D.**

(15.4) Theorem (Deduction theorem in first-order theories). *Suppose $\Gamma, A \vdash_T B$ with a proof in which no formula depending on A is subjected to generalisation on a variable occurring free in A. Then*

$$\Gamma \vdash_T A \Rightarrow B.$$

Proof. Consider a proof of B. If B does not depend on A in the proof, then $\Gamma \vdash B$ and $\Gamma \vdash A \implies B$ using the axiom $B \Rightarrow (A \Rightarrow B)$.

Otherwise, we use induction on proof length. The argument is almost the same as in Sentential Calculus: that is, the first step is A, or from Γ, or a logical or proper axiom of T, or does not depend on A, and where B is derived using MP the same argument applies as in Sentential calculus.

Generalisation makes it different. Suppose C is deduced and later $B = \forall x_i C$, under generalisation, in a step depending on A. By induction,

$$\vdash A \Rightarrow C, \quad \text{so}$$
$$(\forall x_i (A \Rightarrow B)) \quad \text{(Generalisation)}$$

But here the generalisation is applied in a step depending on A, so x_i does not occur free in A and we can use an Axiom V:

$$((\forall x_i(A \Rightarrow C)) \Rightarrow (A \Rightarrow (\forall x_i C))$$

so by Modus Ponens,

$$A \Rightarrow (\forall x_i C)$$

is deduced, i.e., $A \Rightarrow B$. ∎

Remark. The conditions for the Deduction Theorem are automatically met when A is a closed formula (no free variables).