1 A note on Rice's Theorem

1. In defining the partial functions $\phi_m(n) : \mathbb{N} \nrightarrow \mathbb{N}\}$, the length-lex encoding comes up at least twice: in m, and in n.
2. The partial functions $\phi_m()$ are clearly defined, but they are 'slippery.' Almost nothing can be said about them.

   For example, there is no foolproof way to tell, given m and m', whether or not ϕ_m and $\phi_{m'}$ are the same partial function.

1.1 Definition of $\phi_m(n)$, repeated.

We write $\alpha \mapsto n$ where α is a bitstring and n is its length-lex encoding. By 'abuse of notation' we write $n \mapsto \alpha$ to mean that α is the unique bitstring whose length-lex encoding is n.

$\phi_m()$ is a listing of the partial recursive functions. The listing relies on the encoding chosen of Turing machines as bitstrings.

- For each m, suppose $m \mapsto \alpha$. If α encodes a valid Turing machine T, then for any n, suppose that $n \mapsto x$, i.e., x is the unique bitstring length-lex encoded as n.
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  - If T halts on input x, let y be the output of T on input x.
    Then $\phi_m(n)$ is defined as r, where r is the length-lex encoding of y.
    We write $\phi_m(n) = r$, or alternatively $\phi_m(n) \downarrow r$.
  - If T loops on input x, then $\phi_m(n)$ is undefined, i.e., n is not in the domain of ϕ; we write $\phi_m(n) \uparrow$.
- Most natural numbers m do not yield encodings of valid Turing machines. In that case, $\phi_m()$ defined as the partial function whose domain is empty; $\phi_m(n) \uparrow$ for all n.

1.2 Rice's Theorem

We have a complete definition of the partial functions $\phi_m()$, but no useful fact about them can be computed.

That is, if P is a nontrivial fact about ϕ_m, let $Q = \{m : \ \phi_m() \in P\}$. Rice's Theorem says Q is not recursive. That is, there is no Turing machine T such

- If $m \in Q$ then T halts on input x with output 1, where m is the length-lex encoding of x.
- If $m \notin Q$ then T halts on input x with output 0.

This relies crucially on the fact that

- If $m \in Q$ and $\phi_m() = \phi_{m'}()$ then $m' \in Q$.
- In other words: *If m and m' define the same partial function,* then both are in Q or both are outside Q.
- This is because the property that $\phi_m() = \phi_{m'}()$ is well-defined in some sense but is computationally infeasible.

1.3 The set of recursive functions is not recursive

Anyway, $\{m : \phi_m() \text{ is recursive}\}$ is not recursive.