MAU11602 final assignment 2020

Rules and procedures.
0. Check that this document has your name and e-mail address.
1. This is a take-home assignment. Attempt all 4 questions.
2. The assignment is based on the module web notes.
3. You may use the Internet but not post questions to forums. You may email queries to the instructor, but not discuss questions with anyone else.
4. ***Show all work.*** You must show all work. If you do not, you get no credit for the answer, regardless of whether it is correct.
5. Solutions should be scanned or typeset and e-mailed as attachments and submitted ***through the Blackboard system*** by 12 noon on Friday, May 15th.
6. You must conform to the College rules about plagiarism, and you must make a declaration to that effect when you submit your solutions through Blackboard.

Show all work.

Question 1.
1a (8 marks).
Construct a CNF (*not* DNF) with the following truth-table.

X	Y	Z	
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0

X	Y	Z	
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

1b (12 marks).
Construct a resolution proof that the following CNF is inconsistent

$$UWX,\ \ U\overline{W}\,\overline{X},\ \ \overline{U}\,W\overline{X},\ \ \overline{U}\,\overline{W}\,X,\ \ UY,\ \ \overline{U}\,\overline{Y},\ \ \overline{V}\,WY,\ \ VW\overline{Y},$$
$$V\overline{W}\,Y,\ \ \overline{V}\,\overline{W}\,\overline{Y},\ \ VX,\ \ \overline{V}\,\overline{X}$$

Question 2

Consider a first order theory K which has a multiplication operator $\times$, usually omitted so xy means $x \times y$, and has various proper axioms to ensure an equality theory (with $=$ representing equality). It also has an axiom (associativity)

$$\forall x_1 \forall x_2 \forall x_3 (x_1(x_2x_3)) = ((x_1x_2)x_3)$$

Let A', A, B', B be the formulae below

$$A'(x_1) : \forall x_2(x_1x_2 = x_2),\quad A(x_1) : \exists x_1 A'(x_1),\quad B'(x_1) : \forall x_2(x_2x_1 = x_2),\quad B(x_1) : \exists x_1 B'(x_1)$$

and let $K+A$, $K+A+B$ be the theories formed by adding A (respectively A, B) as an axiom (axioms).

2a (8 marks). Show that the interpretation I with domain $\{a, b\}$, $=^I$ the identity relation on $\{a, b\}$, and

$\times^I$	a	b
a	a	b
b	a	b

is a model of $K + A$. (Show this informally without bringing in the idea of 'snapshots.')

2b (4 marks). Determine whether or not it is a model of $K + A + B$.

2c (8 marks). Give a careful proof within K as a first-order theory (with equality: you may use equality reasoning informally) that

$$A'(x_1) \wedge A'(x_3) \wedge B'(x_4) \implies x_1 = x_3.$$

(Hint: the proof should be guided by the fact that $x_1 x_4 = x_1 = x_4$.)

Question 3.

3a (6 marks). Prove that, in any first-order theory K, under a certain condition

$$A(t) \implies \exists x_i A(x_i)$$

is a theorem, stating the required condition.

3b (4 marks). Give an example showing that (3a) is not true (in every interpretation) if that condition is not met.

3c (5 marks). (Note: a countable set D is one which can be listed as a countable sequence $d_1, d_2, \ldots$, so for every nonempty subset X of D there is a smallest i such that $d_i \in X$.)

Suppose that K is a first-order theory, $P(x_1, x_2)$ an atomic formula, and M is a model of K, with countable domain D, such that

$$M \models \exists x_2 P(x_1, x_2)$$

Show that for every $a \in D$ there exists a $b \in D$ such that

$$P^M(a, b)$$

is true.

3d (5 marks) (Continuing 3c).

Show that one can define a function $F : D \to D$ such that for every $a \in D$,

$$M \models P(a, F(a)).$$

(F can be used, for example, in connection with Skolem functions.)

Question 4

Referring to the proof in the notes that it is consistent to assume that infinite integers exist, the idea was to extend PA with a new constant symbol a plus infinitely many formulae which together imply that a is infinite.

4a (6 marks) Construct an interpretation of this extension, with domain $\mathbb{N} \cup \{\infty\}$, extending the definitions of $s(), +, \times$, so that every axiom of PA *except possibly* Axiom 4 (induction), is true in this interpretation. Also, you may ignore the equality axioms.

4a (14 marks) Show that the 7 axioms mentioned in (4a) are true in the interpretation. You may skip the induction axiom.