MAU11602 fifth quiz, week 10
Wed 30/3/22 ANSWERS

This is the last assignment. There will be no Turing program.

Rules and procedures.

1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard.

Question 1. Let $A = \{m : \ \phi_m(n) \downarrow 0\}$ (for all n), $B = \{m : \ \phi_m(n) \downarrow 1\}$. Prove that A and B are recursively inseparable. (Using the fixed point theorem).

Answer. Choose $a \in A$ and $b \in B$. Suppose there is a *recursive* set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

Define a function $f : \mathbb{N} \to \mathbb{N}$:

$$f(n) = \begin{cases} b & \text{if } n \in C \\ a & \text{if } n \in C \end{cases}$$

Since C is recursive, f is recursive.

For any n,

- If $n \in C$, then $f(n) = b \in B$, so $f(n) \neq n$ since $B \cap C = \emptyset$.
- If $n \notin C$, then $f(n) = a \in A$, so $f(n) \neq n$ since $A \subseteq C$.

So f has no fixed point, contradicting the Fixed Point Theorem. So f cannot be recursive and C cannot exist.

Question 2. Rice's Theorem is about recursively enumerable sets rather than partial recursive functions. Recursively enumerable sets are domains of partial recursive functions.

Let R be the set of recursively enumerable sets, so for each $S \in R$, there exists an m such that $S = \{n : \ \phi_m(n) \downarrow\}$. A property P of r.e. sets, i.e., a subset of R, is *trivial* if $P = \emptyset$ or $P = R$. Let P be a nontrivial property. Prove that P is not recursive. . . or, rather that.

$$(A) \quad \{m : \ \{n : \ \phi_m(n) \downarrow\} \in P\}$$

is not recursive, using the Fixed Point Theorem.

This is Rice's Theorem: no nontrivial property of recursively enumerable sets is recursive.

Answer. If P is nontrivial choose a and b so that $a \in (A)$ and $b \notin (A)$. If A were recursive then the function

$$f(n) = \begin{cases} b & \text{if } n \in A \\ a & \text{if } n \notin A \end{cases}$$

would be recursive, and again this contradicts the Fixed Point Theorem. (Details omitted, being much the same as in Question 1.)

Question 3. Answer this question, without too much formality: is the set

$$\{m : m \text{ encodes a TM which has} > 25 \text{ quintuples}\}$$

recursive?

Answer. Yes it is. There is a Turing machine which counts the number of quintuples in the Turing machine encoded by m.

Question 4. Granted, with suitable encodings, that the set of provable formulae in PA is recursively enumerable but not recursive, is the set of unprovable formulae in PA recursively enumerable? Give reasons.

Answer. No it isn't. Otherwise one could build a Turing machine which simulated two machines alternating, the one to show provable, and the other to show unprovable. Then it could decide provability in all cases and the set of provable formulae would be recursive.

Question 5. Let K be the first-order theory with multiplication xy, identity 1, equality axioms, and other proper axioms

$$\text{(i) } x_1(x_2x_3) = (x_1x_2)x_3 \quad \text{and} \quad \text{(ii) } \forall x_1 \exists x_2 (x_1x_2 = 1)$$

Let K' be the theory like K, with a new unary function f, and with (ii) replaced by the axiom

$$\text{(i)}' \ \forall x_1 (x_1 f(x_1) = 1)$$

Given that K is consistent, use the Completeness Theorem to show that K' is a consistent extension of K.

Answer. That K' is an extension of K, we need to prove axiom (ii) in K'. (Replace x by x_1 and y by x_2).

1. $\forall x(xf(x) = 1)$ (given)
2. $xf(x) = 1$ (Ax IV, MP)
3. $xf(x) = 1 \implies \exists y(xy = 1)$ (Ax IV variant)
4. $\exists y(xy = 1)$ (2,3,MP)
5. $\forall x \exists y(xy = 1)$ (4,Gen).

Now let M be a model of K, with domain D: it exists. We need to define f^M.
For any $d \in D$, let σ be any snapshot such that $\sigma_1 = d$. Since

$$M, \sigma \models \exists x_2(x_1x_2 = 1)$$

there exists at least one $e \in D$ such that

$$M, \sigma_{2 \mapsto e} \models (x_1x_2 = 1)$$

Choose one such e and define $f^M(d) = e$. Then

$$M \models \forall x_1(x_1 f(x_1) = 1)$$

This gives a model of K', so K' is consistent. ∎