MAU11602 fourth quiz, week 8, Wed 16/3/22 ANSWERS

Rules and procedures.
1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard.

Question 1. The following set of clauses, from Quiz 2, is inconsistent.

$$\overline{U}VY,\ \ UV\overline{Y},\ \ U\overline{V}Y,\ \ \overline{U}\,\overline{V}\,\overline{Y},\ \ UX,\ \ \overline{U}\,\overline{X},\ \ VWX,\ \ \overline{V}W\overline{X},$$
$$V\overline{W}\,\overline{X},\ \ \overline{V}\,\overline{W}X,\ \ WY,\ \ \overline{W}\,\overline{Y}$$

Every truth-assignment can be represented succinctly as a bitstring, taking the variables $UVWXY$ in that order. For example, the string 10110 means

$$U \mapsto 1, V \mapsto 0, W \mapsto 1, X \mapsto 1, Y \mapsto 0$$

Every truth assignment contradicts at least one clause in the above list. Give one clause contradicting each of the truth-assignments given below.

(i) 01100, (ii) 11011, (iii) 01110, (iv) 10001, (v) 00101

Answer.

$$\text{(i) } 01100 : UX,\quad \text{(ii) } 11011 : \overline{U}\,\overline{V}\,\overline{Y},\quad \text{(iii) } 01110 : U\overline{V}Y,$$
$$\text{(iv) } 10001 : VWX,\quad \text{(v) } 00101 : \overline{W}\,\overline{Y}$$

Question 2. Give a first-order theory K with formulae A and B and an interpretation I such that

$$A \vdash_K B$$

but $A \implies B$ is false in I.
Answer. Constant 0, predicate $=$, A and B: $A : x_1 = 0$, $B : \forall x_1(x_1 = 0)$.

$$x_1 = 0 \vdash \forall x_1(x_1 = 0) \quad \text{(Gen)}$$

Let I be the interpretation with domain $\mathbb{N}$ and $=$ interpreted as equality on $\mathbb{N}$. Let σ be the snapshot $0, 0, 0 \ldots$ (i.e., $\sigma_i = 0$ for all i).

Certainly

$$I, \sigma \models x_1 = 0 : \quad A^\sigma = 1 \text{ (true)}$$

But if $\sigma' = \sigma_{1\mapsto 1}$, then

$$A^{\sigma'} = 0 \text{ (false)}$$

so

$$B^\sigma = 0$$

and

$$A \implies B \quad \text{is false under } \sigma.$$

and therefore $A \implies B$ is false in I. ∎

Question 3. The 'numerals' $\overline{n}$ are important Peano-arithmetic terms defined recursively as follows

$$\overline{0} = 0; \quad \overline{n+1} = s(\overline{n}).$$

So $\overline{3} = s(s(s(0)))$.

Prove (in PA, Peano Arithmetic) that $\overline{2} + \overline{3} = \overline{5}$.

Equality reasoning: if $u = v$ is an axiom or has been proved, you may assume $v = u$ automatically (symmetry) and also assume automatically $u = v, v = w \vdash u = w$.

Answer.

$$\begin{gathered}\overline{2} + \overline{3} = s(s(0)) + s(s(s(0))) = \\ s(s(s(0)) + s(s(0))) = \\ s(s(s(s(0))) + s(0)) = s(s(s(s(s(0))) + 0)) = \\ s(s(s(s(s(0))))) = \overline{5}.\end{gathered}$$

Question 4. Prove 19.9: $x + y = y + x$ by induction on y, using any result up to 19.8. (19.7 may help).

Answer. $P(y) : x + y = y + x$.

$$\begin{gathered}P(0) : x + 0 = 0 + x? \\ 1.\ x + 0 = x \text{ (Pa 5)}; \quad 2.\ 0 + x = x \text{ (19.6)}; \quad 3.\ x + 0 = 0 + x \quad (= \text{transitive}) \\ \text{Induction } 4.\ x + y = y + x \ (P(y) : \text{Hyp}) \\ 5.\ x + s(y) = s(x + y) \quad \text{(Pa 6)} \\ 6.\ s(x + y) = s(y + x) \quad \text{(hyp, substituting equals)} \\ 7.\ s(y + x) = y + s(x) \quad (\text{Pa 6}, = \text{symmetric}) \\ 8.\ y + s(x) = s(y) + x \quad (19.7) \\ 9.\ x + s(y) = s(y) + x \quad (5\text{-}8, = \text{transitive}) : \\ P(s(y)) \quad \blacksquare\end{gathered}$$

Question 5. Define $x \leq y$ to mean $\exists z(x + z = y)$ and $x < y$ to mean $(\exists z \neq 0)(x + z = y)$.

Prove: $x < y \implies s(x) \leq y$.

Answer.

1. $x < y$ (Hyp)
2. $(\exists z \neq 0)(x + z = y)$. Fix z.
3. $z \neq 0 \wedge x + z = y$.
4. $(\exists w)(z = s(w))$. Pa 2. Fix w.
5. $x + s(w) = y$ (substituting equals).
6. $x + s(w) = s(x) + w$ (19.7).

7. $s(x) + w = y$. (Transitivity).
8. $\exists w(s(x) + w = y)$ (Axiom IV, complementary version).
In other words,
9. $x \leq y$.
Since neither w nor z occur *free* in (8), the Fix Rule has been used correctly. Generalisation was not used, so 1 $\implies$ 9 by the Deduction Theorem.