MAU11602 third quiz, week 6
Wed 2/2/22 ANSWERS

Rules and procedures.
1. Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard.

Question 1. $(\exists x_i A)$ is an abbreviation for $(\neg(\forall x_i(\neg A)))$. Give a direct definition of

$$I, \sigma \models \exists x_i A$$

Answer. It is false that for every $d \in D$ (domain of I) $I, \sigma_{i \mapsto d}$ makes A false, so there exists a $d \in D$ (domain of I) such that

$$I, \sigma_{i \mapsto d} \models A$$

Question 2. This question aims to show that Theorem 13.1 is false unless the 't free for x_i' condition is met.

Let $A(x_1)$ be $\exists x_2(x_1 \neq x_2)$ and $t = x_2$, so $A(t)$ is $\exists x_2(x_2 \neq x_2)$.

Let I be an interpretation with domain $\mathbb{N}$ with $=^I$ the equality relation on $\mathbb{N}$.

Let $\sigma = 1, 2, 3, 4 \ldots$ so $\sigma_4 = 4$ and so on.

According to Theorem 13.1, σ' is $\sigma_{1 \mapsto 2}$, since $\tau^\sigma = \sigma_2 = 2$.

Show that one of $A^{\sigma'}, A(t)^\sigma$ is true and the other false, so Theorem 13.1 does not hold.

Answer.

$$A: \ \exists x_2(x_1 \neq x_2).$$

is clearly true under this snapshot σ' (and every snapshot) since all we need is a d such that $\sigma'_{2 \mapsto d} \neq \sigma'_1$, and $\sigma'_1 = 2$, so $d = 3$ will do fine.

$$\exists x_2(x_2 \neq x_2)$$

would be true under σ if and only if for some d, with $\sigma_2 = d$, $d \neq d$, and this is clearly false.

Question 3. Let K be the FOT with one function $+$ and two predicates $=, \leq$. Let I_1 be the interpretation over $\mathbb{N}$, with $=$ interpreted in the usual way, but $\leq$ interpreted indirectly.

$$m \leq^{I_1} n \iff \text{for some } k \text{ in } \mathbb{N},\ m = n + k.$$

Let I_2 be the interpretation like I_1, except that

$$m \leq^{I_2} n \iff \text{for some } k \text{ in } \mathbb{N},\ n = m + k.$$

Verify whether the following is true in either interpretation:

$$\exists x_1 \forall x_2 (x_1 \leq x_2)$$

Answer. The inequality is reversed in I_1 and correct in I_2. The formula says that there is an element which is $\leq$ all others: in I_2, 0 works; in I_1, nothing works since such an element would be infinite.

Question 4. Suppose K is the usual axiom system for group theory, with the formal language $1, f_1, f_2, P_1$ (identity, product, inverse, equality). Briefly, the axioms are associativity, left and right identity, left and right inverse.

We include a third binary function, f_3. We represent it informally as $x_1 \circ x_2$.

$$x_1 \circ x_2 = (x_1 x_2)(x_1^{-1} x_2^{-1})$$

Convert the following, conventional, formulae into the formal language of K:

(i) $x_1 x_2 = x_2 x_1$

(ii) $(x_1 x_2)^{-1} = x_1^{-1} x_2^{-1}$

(iii) $\circ$ is associative.

Answer. (i) $P_1(f_1(x_1, x_2), f_1(x_2, x_1))$

(ii) $P_1(f_2(f_1(x_1, x_2)), f_1(f_2(x_1), f_2(x_2)))$.

(iii) $P_1(f_3(x_1, f_3(x_2, x_3)), f_3(f_3(x_1, x_2), x_3))$

Let D be the collection of all 2×2 invertible real matrices, and I the interpretation with domain D, where f_1^I is matrix product, 1^I is the identity matrix, and f_2^I is matrix inverse. Let σ be the following snapshot.

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 17 & 20 \\ -12 & -14 \end{bmatrix}, \ \sigma_4 = \begin{bmatrix} 18 & 20 \\ -12 & -13 \end{bmatrix},$$

$$\sigma_5 = \begin{bmatrix} -7 & -10 \\ 6 & 8.5 \end{bmatrix}, \ \sigma_6 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_7 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma_8 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \sigma_9 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Values of σ_i for $i \geq 10$ do not affect the calculations.

(iv) Say if the formula $x_3 x_4 = x_4 x_3$ is true under σ.

(v) Say if the formula $x_1 x_6 = x_6 x_1$ is true under σ.

Answer. (iv) Yes. (v) No.

Question 5. And check the following for truth under the snapshot σ:

(i) $x_1 x_5 = x_7$

(ii) $x_5 x_3 = x_7$

(iii) $x_8 \circ (x_9 \circ x_9) = (x_8 \circ x_9) \circ x_9$.

Hint. Invertible matrices A, B commute if and only if $A \circ B = I$ (the identity matrix).

Answer. (i) No. (ii) Yes. (iii) No. According to my calculations,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \circ \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

whereas

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$