Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

SF/JS/SS Maths/TSM **Michaelmas Term 2018**

Mathematics 2361: Computation and Logic

Friday, 14 December 2018 RDS Simmonscourt 9.30–11.30

Prof. Colm Ó Dúnlaing

Instructions to Candidates:

Attempt 3 questions
Please fold and glue the bottom right-hand corner of each answer book, to ensure anonymity.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) (4 marks) Give a Turing machine with input alphabet $\{0, 1, 2\}$ which, on input x, interpreted as an integer to the base 3, halts with $x + 1$ (base 3) on the tape. Neither x nor $x + 1$ need have leading zeroes suppressed. What does the machine do on input λ?

   (b) (6 marks) Describe how to encode all possible Turing machines T (with binary input alphabet) as bitstrings.

   (c) (6 marks) Given

$$\text{HALTING} = \{xy : \ x \text{ encodes a Turing machine which halts on input } y\}$$

   prove that HALTING is not recursive, i.e., its characteristic function cannot be computed by a Turing machine T which halts on all inputs.

   (d) (4 marks) One can show (using a universal Turing machine) that HALTING is recursively enumerable. Is its complement, $\{0, 1\}^* \backslash$HALTING, recursively enumerable? Give a reason.

2. (a) (5 marks) Prove by resolution that the following clauses are inconsistent

$$C\overline{D}, \overline{C}D,$$
$$ABC, A\overline{B}\,\overline{C}, \overline{A}B\overline{C}, \overline{A}\,\overline{B}C,$$
$$AB\overline{D}, A\overline{B}D, \overline{A}BD, \overline{A}\,\overline{B}\,\overline{D}$$

   (b) (4 marks) To 'shortcut' resolution by cancelling two pairs of complementary literals is a serious mistake. Explain why, using UVW and $\overline{V}\,\overline{W}X$ as an example. (The invalid 'resolvent' would be UX.)

   (c) (5 marks) Give a proof in propositional logic (Sentential Calculus, or SC) of the following result.

$$\neg\neg A \vdash_{\text{SC}} A$$

   You may assume $\vdash_{\text{SC}} X \implies X$ for every formula X.

(d) (6 marks) Sketch a proof that if A is a tautology then $\vdash_{\mathrm{SC}} A$. You may assume that the empty clause can be derived from a CNF equivalent to $\neg A$ by repeated resolution, and that if $C \vee L$ and $D \vee \neg L$ are clauses with $C \vee D$ nonempty, then the resolvent $C \vee D$ can be deduced from these clauses in the Sentential Calculus.

3. (a) (4 marks) Given a first-order language, and an interpretation I of that language, define, by induction on the complexity of formulae, the relation

$$I, \sigma \models A$$

where σ is any snapshot.

You need not define the objects $\sigma_{i \mapsto d}$ nor t^σ (the value of t under snapshot σ) where t is a term.

(b) (3 marks) Define what it means for a term t to be free for a variable x_i in a formula $A(x_i)$.

(c) (9 marks) Let $A(x_i)$ be a formula, t a term free for x_i in $A(x_i)$, I an interpretation, σ a snapshot, and $\tau = \sigma_{i \mapsto t^\sigma}$. Then one can prove, by induction on the complexity of A, that

$$I, \sigma \models A(t) \quad \text{if and only if} \quad I, \tau \models A(x_i)$$

Prove two cases: (i) where A is an atomic formula, and (ii) where $A(x_i)$ has the form $\forall x_j B(x_i, x_j)$, where $x_j \neq x_i$ and x_i occurs free in $B(x_i, x_j)$.

You may assume without proof that if x_j does not occur in t then t^σ is independent of σ_j, and if $t_r(x_i)$ is any term then $t_r(x_i)^\tau = t_r(t)^\sigma$.

(d) (4 marks) Give a formula $A(x_1)$ with just x_1 free, and a suitable interpretation I such that $I \models A(x_1)$ but not $I \models A(x_2)$.

4. (a) (6 marks) Define when two sets X, Z of natural numbers are *recursively inseparable.* Using the Fixed Point Theorem (without proving it), show that

$$C_0 = \{m : \ \phi_m(0) \downarrow 0\} \quad \text{and} \quad C_1 = \{m : \ \phi_m(0) \downarrow 1\}$$

are recursively inseparable.

(b) (8 marks) Recall the existence of a primitive recursive relation

$$\mathrm{Result}\,(m, n, y, S)$$

to describe Turing machine computations, with a corresponding formula $\mathrm{Result}\,(x_1, x_2, x_3, x_4)$ of Peano Arithmetic (PA) expressing that relation.

(i) Give a formula of PA expressing the relation

$$\phi_m(n) \downarrow y$$

(ii) Deduce that the set XX of theorems of PA, and the set ZZ of formulae of PA false in $\mathbb{N}$, are recursively inseparable.

(c) (6 marks) Deduce that XX, ZZ, and TT, the set of formulae *true* in $\mathbb{N}$, are not recursive.