7 Bases

Except inside computers, nonnegative integers are generally written to base 10. In computer programming, and in computers, numbers to base (sometimes called radix) 2, 8, or 16, can be useful. In general, given a radix $R > 1$, and a nonnegative integer x, x to base R is a sequence, with $k > 0$,

$$a_{k-1}a_{k-2}\ldots a_1a_0$$

such that

$$0 \le a_j \le R-1, \quad \text{for } 0 \le j < k$$

and

$$a_{k-1}R^{k-1} + a_{k-2}R^{k-2} + \ldots + a_1R + a_0 = x$$

We call it *leading zero suppressed* if either $k = 1$ or $k > 1$ and $a_{k-1} > 0$.

Hand calculation: Horner's method. Given R and $a_{k-1}\ldots a_0$, build up x as follows. The last value of y will be the answer.

$$\begin{gathered} y = a_{k-1} \\ \text{if } k > 1, y = yR + a_{k-2} \\ \text{if } k > 2, y = yR + a_{k-3} \\ \text{and so on until}\ldots y = yR + a_0 \end{gathered}$$

The final value of y gives the number x.

To avoid ambiguity, we can write $(a_{k-1}\ldots a_0)_R$ to show that the radix is R.

Example. $(1100011)_2$. Evaluate.

$$\begin{gathered} y = 1 \\ y = 2 \times 1 + 1 = 3 \\ y = 2 \times y + 0 = 6 \\ y = 2 \times y + 0 = 12 \\ y = 2 \times y + 0 = 24 \\ y = 2 \times y + 1 = 49 \\ y = 2 \times y + 1 = 99 \end{gathered}$$

Answer: 99.

Example. $(143)_8$. Evaluate.

$$\begin{gathered} y = 1 \\ y = 8 \times y + r = 12 \\ y = 8 \times 12 + 3 = 99. \end{gathered}$$

Answer: 99.

Example. $(63)_{16}$. Evaluate.

$$y = 6$$
$$y = 16 \times 6 + 3 = 99.$$

Answer: 99.

So: *converting to decimal is a matter of repeatedly multiplying by R and adding another digit, from left to right.*

Converting to radix R is a matter of repeatedly taking remainder modulo R (producing another digit) and dividing by R, *from right to left.*

Convert 99 to binary.

$$99 \mod 2 = 1(a_0); 99 \div 2 = 49; \qquad 49 \mod 2 = 1(a_1); 49 \div 2 = 24$$
$$24 \mod 2 = 0(a_2); 24 \div 2 = 12; \qquad 12 \mod 2 = 0(a_3); 12 \div 2 = 6$$
$$6 \mod 2 = 0(a_4); 6 \div 2 = 3; \qquad 3 \mod 2 = 1(a_5); 3 \div 2 = 1$$
$$1 \mod 2 = 1(a_6); 1 \div 2 = 0$$

Any further calculations will only produce leading zeroes. So: $99 = (1100011)_2$.

Convert 99 to octal (base 8)

$$99 \mod 8 = 3(a_0); 99 \div 8 = 12; \qquad 12 \mod 8 = 4(a_1); 12 \div 8 = 1$$
$$1 \mod 8 = 1(a_3); \qquad 1 \div 8 = 0$$

Answer: $99 = (143)_8$.

Convert 99 to what is called ***hexadecimal*** (base 16).

$$99 \mod 16 = 3(a_0); 99 \div 16 = 6; \qquad 6 \mod 16 = 6(a_1); 6 \div 16 = 0$$

Answer: $99 = (63)_{16}$.

Convert 15 to hexadecimal.

$$15 \mod 16 = 15; \qquad 15 \div 16 = 0$$

Answer: ??. You see, 15 should be a single hex digit. **Solution** The 10 decimal digits are extended to 16 **hex digits**.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f

Question: Does that mean that octal and hexadecimal numbers are also found in computers?

Answer. Not exactly. All data is encoded in binary. Octal and hexadecimal are used to give binary numbers in compact form. This is because it is very easy to convert between octal and binary and between hexadecimal and binary. Every 3 binary digits give you one octal digit. Every 4 binary digits give you one hexadecimal digit. Convert binary to octal or hexadecimal working from right to left.

Example. Convert $(1100011)_2$ to octal.

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Answer. Form groups of 3 from right to left.
       1 100 011
       1   4   3:    143 in octal
```

Convert $(1100011)_2$ to hexadecimal.

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Answer.  Form groups of 4 from right to left
 110 0011
   6    3  : 63 in hexadecimal.
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How does one convert 4 binary digits into a decimal number or directly to a hex digit? Either do the calculation directly:

$$110 : y = 1; y = 2 \times 1 + 1 = 3; y = 2 \times 3 + 0 = 6$$

or construct a table

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binary  decimal  hex         binary  decimal  hex
0000        0     0          1000        8     8
0001        1     1          1001        9     9
0010        2     2          1010       10     a
....etcetera                 1011       11     b etcetera
....
0111        7     7          1111       15     f
```

Now convert 15 to hex. Answer: $15 = (f)_{16}$.