Deciding the decidable

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Feel free to ask questions.

- Meaning of title
- Turing
- Turing
- Turing
- Thue
- Thue
- Thue
- Markov

Context

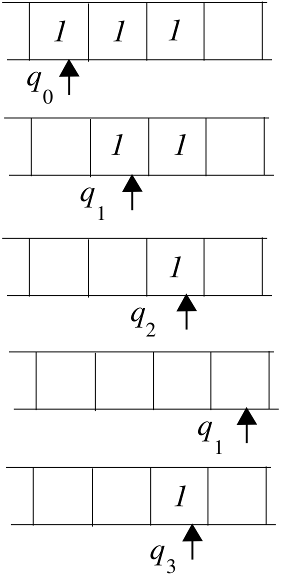
What does the title mean? Certain problems, such as Hilbert's 10th problem, to design an algorithm which solves all diophantine equations, have no solution (Davis, Putnam, Julia Robinson, Matiyasevich, finished around 1970).

The point is that some questions are (computationally) undecidable/unsolvable; there is no algorithm to decide them. Now the title 'deciding the decidable' may suggest coming up with algorithms solving many specific instances. But that's much too difficult. What we aim at is the following result:

Decidability is undecidable.

Turing machines

A Turing machine is a very simple computer, and this figure illustrates a Turing machine and a ‘computation.’



$q_0\, 1\, B\, R\, q_1$

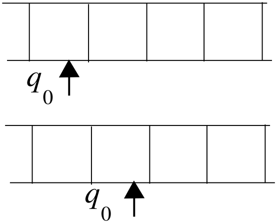
$q_1\, 1\, B\, R\, q_2$

$q_1\, B\, 1\, L\, q_3$

$q_2\, 1\, B\, R\, q_1$

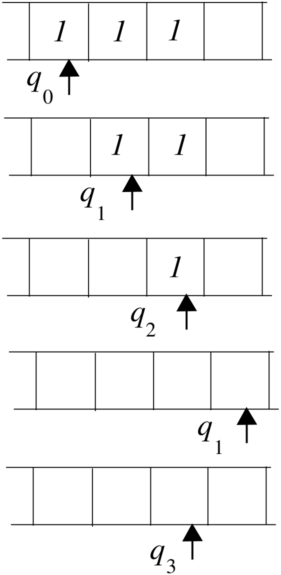
$q_2\, B\, 0\, L\, q_3$

$q_0\, B\, B\, R\, q_0$



Turing machines

This Turing machine is like the previous one, except that when it halts it halts with blank tape.



$q_0 1\ B\ R\ q_1$

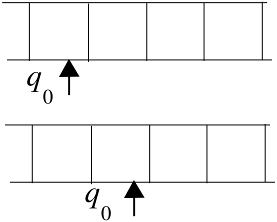
$q_1 1\ B\ R\ q_2$

$q_1 B\ B\ L\ q_3$ *

$q_2 1\ B\ R\ q_1$

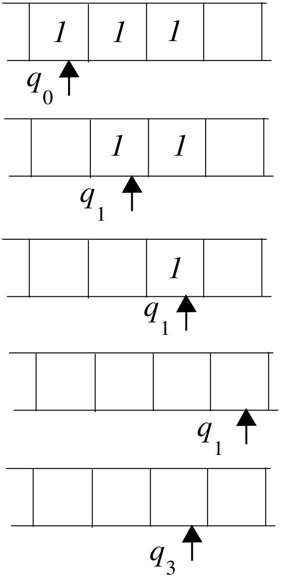
$q_2 B\ B\ L\ q_3$ *

$q_0 B\ B\ R\ q_0$



Turing machines

This Turing machine is a simplified form of the previous one.

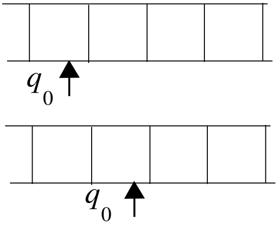


$q_0 1 B R q_1$

$q_1 1 B R q_1$

$q_1 B B L q_3$

$q_0 B B R q_0$



A computation shown with strings

Definition Let Σ be a set. Any finite sequence of elements from Σ is called a *string*. Below, *strings of symbols* show the nonblank part of the tape in each step of the computation.

$$\$q_0 111¢ \quad \$Bq_1 11¢ \quad \$BBq_1 1¢ \quad \$BBBq_1¢ \quad \$BBq_3 B¢$$

Definition. A *Thue system over* Σ is a finite set of pairs of strings over Σ. The pairs are normally shown in the notation $\alpha \leftrightarrow \beta$. A Thue system operates on strings in the following way:
Suppose $u\alpha v$ is a string where a pair $\alpha \leftrightarrow \beta$ or $\beta \leftrightarrow \alpha$ is in T.
Then $u\alpha v \leftrightarrow_T u\beta v$.
Given strings x, y, if there exists a sequence of strings:

$$x = x_1 \leftrightarrow_T x_2 \ldots \leftrightarrow_T x_k = y$$

then we say x, y are *congruent modulo* T and write

$$x {\leftrightarrow_T}^* y$$

Turing machines and Thue systems

Given a Turing machine M, computation of M can be represented by a sequence of strings and the steps in the computation by rules of a particular Thue system T constructed from M.

$$\$q_0 111¢ \quad \$Bq_1 11¢ \quad \$BBq_1 1¢ \quad \$BBBq_1 ¢ \quad \$BBq_3 B¢$$

The rough idea is: if $q_i XYRq_j$ — this is called a *quintuple* — is part of the Turing machine, then

$$q_i X \leftrightarrow Yq_j$$

implements the quintuple, sort of.

To make things work, one needs to introduce dummy symbols and other bits and pieces. But finally one can achieve this result:

Proposition If M is a Turing machine, then one can construct a Thue system T over a certain alphabet so that for any input string x for M, *M halts on input x* if and only if

$$\$q_0 x¢ \leftrightarrow_T^* \$¢$$

Halting problem

- This is significant because there is no algorithm to decide in general when M halts on input x.
- The *word problem* for a Thue system T over Σ is to decide, given two strings x and y, whether or not $x \leftrightarrow_T^* y$.
- **Corollary.** There exists a Thue system whose word problem is undecidable.

Definition. A Thue system T over an alphabet Σ is *coarsest* if $x {\leftrightarrow_T}^* y$ for all strings x and y over Σ.
The *empty* string is denoted λ.
Let T be a Thue system whose word problem is undecidable. Let Σ be the alphabet of T and choose symbols $c, d \notin \Sigma$.
Extend T to a larger Thue system $T_{G,H}$ over $\Sigma \cup \{c, d\}$ by adding the following rules

$$cGd \leftrightarrow \lambda$$
$$xcHd \leftrightarrow cHd, \quad x \in \Sigma \cup \{c, d\}$$

A result due to Markov.

Proposition. (i) If $G \leftrightarrow_T{}^* H$, then $T_{G,H}$ is coarsest (with alphabet $\Sigma \cup \{c, d\}$).
(ii) If $G \leftrightarrow_T{}^* H$, then for any strings $x, y \in \Sigma^*$, not containing c or d, if $x \leftrightarrow_{T_{G,H}}{}^* y$, then $x \leftrightarrow_T{}^* y$.
Proof. (i) is easy. For (ii), one observes that c, d behave like 'matching parentheses' and there is a notion of nesting depth which allows parts of the strings to be separated out.

Markov, conclusion

Corollary It is undecidable of a Thue system whether the word problem is decidable. Decidability is undecidable.

Proof. In case (i), the word problem for $T_{G,H}$ is trivially decidable.
In case (ii), the question, given strings x, y over Σ, of whether $x \leftrightarrow_T * y$ is the same as whether $x \leftrightarrow_{T_{G,H}}{}^* y$, so the word problem for $T_{G,H}$ is also undecidable.
Therefore $x \leftrightarrow_T{}^* y$ if and only if the word problem for $T_{G,H}$ is decidable. Q.E.D.

Decidability is undecidable.