Do sets exist?

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This is an informal review of 20th century solutions to Cantor's Continuum Hypothesis, paying attention to the 'formalist' position, namely, that the existence of sets is irrelevant, — i.e., the only requirement is that Set Theory be consistent.

Breakdown:

- The formalist position, as described by Paul Cohen.
- Some number theory and the Heap paradox of Eubulides.
- Cantor's Set Theory, with its different orders of infinity, Cantor's paradox, and the Continuum Hypothesis (CH).
- Gödel's construction of 'makeshift models' (my phrase). This has something to say to the formalists.
- Inner models and Gödel's proof that CH is consistent with Set Theory.
- The minimal model, forcing, and Cohen's proof that CH is independent of Set Theory.
- Does Leopold Bloom exist?

The formalists, according to Cohen

This talk is concerned with the formalist approach to mathematics, and its opposite, call it Idealist or Platonist or whatever. They are described in Paul Cohen's classical book on the Continuum Hypothesis. The formalists say that

> *...mathematics should be regarded as a purely formal game played with marks on paper, and the only requirement this game need fulfil is that it does not lead to an inconsistency.*

This defensiveness was prompted by the discovery of internal inconsistencies, such as Cantor's Paradox (given later). When Cohen added that

> *...most mathematicians are more or less idealists in their view that sets actually exist,*

he was describing himself.

There is little more to say explicitly about the formalist/idealist division. But it is *implicit* in what follows.

Proofs are mentioned, and even shown, and it is clear that they are much more than a game played with marks on paper.

Some Number theory

This is about properties of natural numbers. The set $\mathbb{N}$ of **Natural Numbers** is

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

It is ALWAYS denoted $\mathbb{N}$.

Properties of natural numbers can be proved using **mathematical induction.** To prove that a property $P(n)$ is true for every $n \in \mathbb{N}$, it is enough to prove two things:

- $P(0)$, i.e., Zero has this property, and
- $$P(n) \implies P(n+1):$$

  whenever n has the property, so has $n+1$.

Example. For every n,

$$n^2 = 1 + 3 + 5 + \ldots + (2n-1)$$

that is, n^2 equals the sum of the first n *odd* numbers.

- 0^2 is the sum of the first 0 odd numbers.
- Assuming n^2 is the sum of the first n odd numbers, the sum of the first $n+1$ is

  $$1 + 3 + \ldots + 2n - 1 + 2n + 1$$
  $$= n^2 + 2n + 1 = (n+1)^2.$$

The heap paradox

Eubulides used mathematical induction to show that a heap of sand cannot exist.

- A single grain of sand is not a heap.
- If n grains do not make a heap, adding one grain doesn't create a heap.

Obviously, the natural number system does not say interesting things about heaps of sand. I would say that a heap of sand contains, *for practical purposes,* infinitely many grains.

This is a good excuse to admit infinite numbers.

Infinity is not always mysterious.

Infinite numbers are often easier to work with.

Cantor's Set Theory

Cantor invented a theory of sets in which infinite quantities occur naturally. According to him, 'a set is a collection of things intuitively thought of as a whole.'

He invented ways in which sets could be considered to be of equal size (cardinality), even if they were infinite.

Notation for cardinality:

$$|X|$$

is the cardinality of a set X. There is a **smallest infinite** cardinal, which happens to be the cardinality of the natural numbers. He called in $\aleph_0$.

$$\aleph_0 = |\mathbb{N}|$$

Sets, such as the set of natural numbers, or the set of integers, with this cardinality, are called *countably infinite*. Next

$$\aleph_1$$

is the smallest cardinality bigger than $\aleph_0$, and so on...

Cardinality of the power set

The set of all subsets of a set X is written

$$\mathcal{P}X$$

For example,

$$\mathcal{P}\{a,b\} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

If $|X| = n$ is finite then $|\mathcal{P}X| = 2^n$ is bigger than X. Cantor proved this for every infinite set X as well.

Theorem (Cantor) $|X| < |\mathcal{P}X|$ always.
Proof.

- Suppose that you had some scheme which matched with every $x \in X$ a subset $f(x)$ of X.
- We must show that not every subset of X is matched with an element of X.
- One can argue very simply that the set

  $$S = \{x \in X : \quad x \notin f(x)\}$$

  is not matched to any $x \in X$. **Q.E.D.**

Cantor's Paradox is that

'the set of all sets'

call it U, contains $\mathcal{P}U$. It is absurd that $|U| < |\mathcal{P}U|$.

Cantor's Continuum Hypothesis (CH)

$\mathbb{R}$ is the 'real number continuum,' including all negative numbers, $\sqrt{2}$, π, and so on.

It can be proved that the number of subsets of $\mathbb{N}$ exactly matches the number of real numbers, i.e.,

$$|\mathbb{R}| = |\mathcal{P}\mathbb{N}|.$$

By Cantor's Theorem,

$$|\mathbb{N}| < |\mathcal{P}\mathbb{N}|$$

So $\mathcal{P}\mathbb{N}$ is not countably infinite; therefore $\mathbb{R}$ is not countably infinite.

Since $|\mathbb{N}| = \aleph_0$ and $|\mathcal{P}\mathbb{N}| = |\mathbb{R}|$,

$$\aleph_0 < |\mathbb{R}|.$$

Recall that $\aleph_1$ is the smallest cardinality bigger than $\aleph_0$. It follows that

$$\aleph_1 \leq |\mathbb{R}|.$$

Cantor's Continuum Hypothesis (CH) is

$$|\mathbb{R}| = \aleph_1$$

Makeshift models

After about 50 years of effort by various mathematicians to prove or disprove CH, Gödel showed in 1940 that CH is *relatively consistent.* The argument is based on what I call 'makeshift models.'

Theories have 'models' in the sense that $\mathbb{N}$ is a 'model' of number theory.

Theorem (Gödel, c. 1930). Every consistent theory has a countably infinite model.

His argument is based on

> If you cannot **disprove** a statement A in a theory T, then $T + A$ (theory T plus assumption of A) is consistent.

You keep going for countably many steps, adding more and more assumptions, so they are all consistent. Eventually all the assumptions made add up to a full description of a model (of the original theory). $\blacksquare$

The model can be rather unnatural, as the only rule is to avoid inconsistencies. Hence I call it a 'makeshift model.'

The formalist position is weakened: the formalist is playing a game with marks on paper. But if the game does not lead to inconsistencies, then there does exist something whose properties are being deduced. If it does, then the formalist is wasting everybody's time.

Paradox: if **Set Theory** is consistent (and it probably is!) then there is a *countable* (makeshift) model of set theory.

Relative consistency of CH

Gödel proved that the Continuum Hypothesis CH is *relatively consistent.*

Idea: **suppose** that Set Theory is consistent.

- It has a 'makeshift model' V
- Gödel discovered how to extract a 'submodel' L of 'constructible' sets which is a model for set theory **plus** CH.

(The details are extremely complicated.)

Independence of CH

Cohen proved that the negation of CH is *relatively consistent.*

He began by assuming there exists a **minimal standard model** M for set theory. This is a huge assumption. It is much more than a makeshift model, same as $\mathbb{N}$ isn't any old model of number theory.

His justification was

> *all our intuition comes from our belief in the natural, almost physical, model of the mathematical universe.*

He wanted to transform M into a *standard* model N in which CH is false. His method he called 'forcing.' He wanted to be able to construct conditions which 'forced' certain things to happen in N. He was successful, and indeed could construct standard models in which $|\mathbb{R}|$ could be almost anything: for example,

$$|\mathbb{R}| = \aleph_{17}$$

is relatively consistent. So much for the overview. This is not easy!

Martin Davis always insisted that Cohen's forcing method was not something you got a feel for. If you want to learn it you have to sit down and work through all the details. I must confess I have yet to do so, but here's an extract from his book ...

Sampler from Cohen's book...

8. P forces $c_1 = c_2$, where $c_1 \in S_\alpha$, $c_2 \in S_\beta$, $\gamma = \max(\alpha, \beta)$, $\alpha < \beta$, if either $\gamma = 0$ and $c_1 = c_2$ as members of S_0, or $\gamma > 0$ and P forces $\forall_\gamma x(x \in c_1 \Leftrightarrow x \in c_2)$.

9. P forces $c_1 \in c_2$ where $c_1 \in S_\alpha$, $c_2 \in S_\beta$, $\alpha < \beta$, if P forces $A(c_1)$ where $A(x) = \phi_\beta(c_2)$.

10. P forces $c_1 \in c_2$, where $c_1 \in S_\alpha$, $c_2 \in S_\beta$, $\alpha \geq \beta$ and $\alpha \neq 0$, if for some $c_3 \in S_\gamma$ where $\gamma = 0$ if $\beta = 0$ and otherwise $\gamma < \beta$, P forces $\forall_\alpha x(x \in c_1 \iff x \in c_2)\&(c_3 \in c_2)$.

11. P forces $c_1 \in c_2$, where $c_1, c_2 \in S_0$ if (...various possibilities or ...) P explicitly mentions the condition $c_1 \in c_2$.

Existence of Leopold Bloom

Paul Cohen was credited with having an uncanny intuition about Set Theory.[a] Set theory is the creation of Cantor, so Cohen had an uncanny intuition about Cantor's ideas.

Perhaps Cantor's ideas were so readily adopted because Cantor had a knack for simplicity and clear description, which meant his creation became common property. In this light, Cohen's remark that

> *all our intuition comes from our belief in the natural, almost physical, model of the mathematical universe*

is not quite so mystical. We conclude with a literary analogy.

In 1982 a plaque was erected in Clanbrassil Street to commemorate its famous inhabitant, Leopold Bloom, born 1866. Neighbours did not remember him living there, not surprisingly, as our only information about him is through James Joyce's *Ulysses*. However, his character, habits, interests, experiences, etcetera, are clearly described, and when people discuss Leopold Bloom they have no doubt that they know him well.

He is consistent; therefore, perhaps, created by Mr. Joyce, although of course he never *existed*, he continues to exist in people's minds.

[a] and a great deal more.