2 Zermelo-Fraenkel axioms for set theory

ZF is a first-order theory with two binary predicates, $=$ and $\in$, and several predicates and functions introduced as abbreviations.

The following axiom can be counted as a 'logical axiom;' it is the only instance of an equality axiom that we need to ensure ZF is an equality theory:

$$\forall x_1 \forall x_2 \forall x_3 (x_1 = x_2 \implies (x_1 \in x_3 \iff x_2 \in x_3)) \tag{2.1}$$

The first proper axiom is EXTENSIONALITY: two objects (sets) are equal if and only if they have the same members. Note that as a consequence of the extensionality axioms, all objects are sets; there are no 'atoms.'

$$\forall x_1 \forall x_2 [x_1 = x_2 \iff \forall x_3 (x_3 \in x_1 \iff x_3 \in x_2)] \tag{2.2}$$

The next axiom can be deduced, in a contorted way, from the other axioms, but it's simpler to make it explicit: it is the axiom of the null set.

$$\exists x_1 \forall x_2 (x_2 \notin x_1) \tag{2.3}$$

It is easily proved that this set x_1 — the empty set — is unique. We introduce the symbol $\varnothing$ to represent this. Rather than introducing it as a symbol of ZF, we regard it as shorthand. If a formula A involves this symbol, it can be regarded as shorthand for a formula

$$\exists x_k [A' \wedge \forall x_1 (x_1 \notin x_k)],$$

where $k \neq 1$ and x_k does not occur free in A, and A' is obtained by replacing every occurrence of $\varnothing$ by x_k in the formula A.

The next axiom, the union axiom, is as follows:

$$\forall x_1 \exists x_2 \forall x_3 (x_3 \in x_2 \iff \exists x_4 (x_4 \in x_1 \ \wedge \ x_3 \in x_4)) \tag{2.4}$$

Again, it is easy to prove that for any set x_1 the union set x_2 is unique. We use $\bigcup x_1$ to denote the union of (all members of) x_1. Again, this is a shorthand, not part of the language, and a formula A involving $\bigcup x_1$, say, can be translated into a formula

$$\exists x_k [\forall x_2 (x_2 \in x_k \iff \exists x_3 (x_3 \in x_1 \wedge x_2 \in x_3)) \wedge A']$$

where x_k differs from x_2 and does not occur free in A, and A' is obtained by replacing every occurrence of $\bigcup x_1$ by x_k in A. Repeating this process as often as needed, we translate A into a formula of ZF.

(2.1) Definition $x_1 \subseteq x_2$ *abbreviates* $\forall x_3 (x_3 \in x_1 \implies x_3 \in x_2)$.

The next axiom is the axiom of the power set:

$$\forall x_1 \exists x_2 \forall x_3 (x_3 \in x_2 \iff x_3 \subseteq x_1) \tag{2.5}$$

We use $\mathcal{P}x_1$ to denote this set x_2, which again depends uniquely on x_1. The symbol $\mathcal{P}$ is implicitly introduced by definition.

The last axiom, for the time being — axioms of infinity and well-foundedness will be introduced later — is the REPLACEMENT AXIOM. It says that if $A(x_i, x_j)$ is a formula of ZF having the indicated variables (possibly) as free variables, and perhaps some more, then the closure of the following formula

$$(\forall x_i \exists^* x_j A(x_i, x_j)) \Rightarrow (\forall x_p \exists x_q \forall x_j (x_j \in x_q \iff \exists x_i (x_i \in x_p \wedge A(x_i, x_j)))) \tag{2.6}$$

is an instance of the Replacement axiom scheme. What it says is that if the formula $A(x_i, x_j)$ defines a 'partial function,' then for any set x_p its 'image' x_q under this 'function' is also a set. The variables x_p and x_q are chosen not to occur free in A and to differ from x_i and x_j.

One should note that the replacement axiom is in fact a scheme of countably many formulae of ZF. A consequence of the replacement axiom is the (slightly weaker) *comprehension principle,*[1] which says that for any set x_p and property $B(x_i)$ (defined by a formula of ZF), the subset of elements of x_p satisfying $B(x_i)$ is also a set.

$$\forall x_p \exists x_q \forall x_i (x_i \in x_q \iff (x_i \in x_p \wedge B(x_i))) \tag{2.7}$$

The comprehension principle is easily deduced from the Replacement axiom.

(2.2) Notation *We shall use the conventional notation*

$$\{x_i \in x_p \colon B(x_i)\}$$

to denote the above-mentioned subset x_q.

This notation is not formally part of the language, but it is a convenient and redundant notation. The set x_q whose existence is given by the comprehension principle is easily proved unique.

Again,

$$\{x\} \equiv \{z \in \mathcal{P}(x) \colon z = x\}.$$

The 'axiom' of the pair set.

$$\forall u \forall v \exists w \forall x (x \in w \iff (x = u \vee x = v)).$$

The set w is unique by the extensionality axiom; one writes $w = \{u, v\}$ in the usual way. This 'axiom' can be deduced from the replacement axiom in the following way: first, show that the set $\{\varnothing, \{\varnothing\}\}$ is well-defined with two distinct elements (this set is a subset of $\mathcal{P}\mathcal{P}\varnothing$, and can be defined using the principle of comprehension). Then let $\mathcal{A}(x_1, x_2, u, v)$ represent the formula

$$(x_1 = \varnothing \ \wedge \ x_2 = u) \vee (x_1 = \{\varnothing\} \ \wedge \ x_2 = v)$$

Then $\{u, v\}$ is the range of $\{\varnothing, \{\varnothing\}\}$ under the functional relation $\mathcal{A}$.

[1] Zermelo's axioms had the comprehension principle rather than the replacement axiom; Fraenkel strengthened his axioms by contributing the replacement principle. I have the impression that Zermelo's theory is strictly weaker than ZF.

From now on, to avoid excessive use of subscripts, we shall use $u, v, w \ldots$ as suitable symbols for variables. Implicitly these are to be chosen from the list $x_1, \ldots$.

Set intersection and difference. If x is nonempty, $\bigcap x \equiv \{y \in \bigcup x \colon (\forall z \in x)(y \in z)\}$; $u \cup v \equiv \bigcup\{u, v\}$; $u \cap v \equiv \bigcap\{u, v\}$; $u \backslash v \equiv \{x \in u \colon x \notin v\}$.

Ordered pairs. One defines the ordered pair (u, v) as $\{\{u\}, \{u, v\}\}$. Clearly, u, the first element of the ordered pair, can be characterised as the unique element belonging to $\bigcap(u, v)$; the second element can be identified as the first (if (u, v) is a singleton set), otherwise the other element of $\bigcup(u, v)$. Two ordered pairs are equal if and only if their first and second elements are respectively equal.

In general, for any finite $n > 2$, one defines $(x_1, \ldots, x_n)$ recursively as $(x_1, (x_2, \ldots x_n))$.

The **Cartesian product** $u \times v$ is defined as $\{z \in \mathcal{PP}(u \cup v) \colon (\exists x \in u)(\exists y \in v)(z = (x, y))\}$. Again, this notation is informal and not part of ZF.

Functions. $f \colon x \to y$, f is a function from x into y, abbreviates the formula

$$f \subseteq x \times y \ \wedge \ (\forall u \in x)(\exists! v \in y)((u, v) \in f)$$

Collections. We use the notation

$$\{x \colon \mathcal{A}(x)\}$$

informally to denote the 'collection' of objects x having property $\mathcal{A}$ ($\mathcal{A}$ is any formula of ZF). Such an object may be too 'big,' in some sense, to be a set; that is, not every formula of ZF need determine a set. The classical example is

$$\{x \colon x \notin x\} \tag{2.8}$$

(Russell's Paradox.) For if the above formula describes a set, then in more precise language

$$(\exists y)(\forall x)(x \in y \iff x \notin x).$$

'Fix y.' Then one easily shows

$$y \in y \iff y \notin y,$$

which is a contradiction, hence such a set y does not exist (this is a theorem of ZF).

The 'universe,' the 'collection of all sets,' is likewise not a set, because if it were, then by the comprehension principle, Russell's paradoxical collection would also be a set.

Relations. A formula $\mathcal{A}$ of ZF with free variables $x_1, \ldots x_n$, say, can be considered to define a relation as a collection of n-tuples

$$\{(x_1, \ldots x_n) \colon \mathcal{A}(x_1, \ldots x_n)\}$$

where this collection is not necessarily a set.

(2.8) Remark *Certain 'facts about sets' will be proved in the conventional mathematical style. Unless otherwise stated, these facts are theorems of ZF and could be proved laboriously within the formal theory.*