3 Ordinals

A relation $u < v$ (defined by a formula of ZF with u, v free) WELL-ORDERS a set x if the following hold:

TRANSITIVITY $<$ is transitive on x, i.e., for all u, v, w in x, if $u < v$ and $v < w$ then $u < w$

TRICHOTOMY For all $u, v \in x$, $u < v \;\vee\; u = v \;\vee\; v < u$.

WELL-FOUNDED. Every nonempty subset y of x has a least element with respect to $<$:

$$(\forall y \subseteq x)(y \neq \varnothing \implies (\exists z \in y)(\forall w \in y)(w \not< z).)$$

(3.2) Lemma *If $<$ well-orders x and $y \subseteq x$ then $<$ well-orders y (trivial).* ∎

(3.3) Definition *Given $u, v \in x$, write*

- *$u \leqslant v$ if $u < v \vee u = v$*
- *$u > v$ if $v < u$*
- *$u \geqslant v$ if $u > v \vee u = v$.*

Remark. Let y be a nonempty subset of a well-ordered set x. There exists an element t of y with the property that for every $v \in y$, $v \not< t$. From the trichotomy

$$(\forall v \in y)(v \geqslant t)$$

(3.4) Lemma *The trichotomy is strict: that is for any $u, v \in x$, exactly one of the following holds:* (i) *$u < v$;* (ii) *$u = v$;* (iii) *$v < u$.*

Proof. (i) For any $u \in x$, $\{u\}$ has an element t such that $v \not< t$ for every $v \in \{u\}$. So $t = u$ and therefore $u \not< u$; i.e., if $u = v$ then $u \not< v$.

Counterpositive: if $u < v$, both in x, then $u \neq v$.

(ii) If $u < v$ then $v \not< u$, because otherwise $u < u$ by transitivity.

So: if $u < v$ then $u \neq v$ and $v \not< u$.

(iii) By symmetry, if $v < u$ then $v \neq u$ and $u \not< v$.

(iv) If $u = v$ then $u \not< v$ by (ii) and $v \not< u$ by (iii). ∎

Remark For any $u, v \in x$, either $u < v$ or $u \geqslant v$ and not both. And so on.

(3.5) Definition *An* initial segment *s of a set x well-ordered by $<$ is a subset such that*

$$(\forall u, v \in x)((v \in s \;\wedge u < v) \implies u \in s)$$

Equivalently (rather trivially)

$$(v \in s \;\wedge u \leqslant v) \implies u \in s.$$

(3.6) Lemma *The initial segments of x are x itself and all subsets of the form $\{z \in x \colon z < t\}$ for some element t of x.*

Proof. Let s be an initial segment of x, and suppose $s \neq x$; in this case, the set difference $x \backslash s$ is nonempty, and since $<$ is well-founded on x, $x \backslash s$ must have a least element t.

Note that $t \notin s$.

Therefore, for every $w \in x$,

- if $w \notin s$ then $w \geqslant t$, and
- if $w \in s$ then $w < t$, for otherwise $t \leqslant w$ and, since s is an initial segment, $t \in s$ which is false.

Therefore $s = \{w \in x : \ w < t\}$ as required. ▮

(3.7) Definition *A set x is* TRANSITIVE *if, for all $y \in x$, $y \subseteq x$. (This concept is of technical importance in ZF set theory.)*

(3.8) Definition *An* ORDINAL *is a transitive set which is well-ordered by the elementhood relation $\in$. This property is easily expressed as a formula of* ZF*, which we abbreviate as* $\mathrm{On}(x)$ *('x is an ordinal.')*

(3.9) Notation *Small greek letters early in the Greek alphabet: $\alpha, \ldots, \theta$, will be used to denote variables 'restricted to the collection of ordinals.' Thus $\forall \alpha \ldots$ abbreviates $\forall u(\mathrm{On}(u) \implies \ldots)$, and so on.*

Write $\alpha \leqslant \beta$ if $\alpha \in \beta \vee \alpha = \beta$.

(3.10) Lemma $\forall \alpha (\forall u \in \alpha) \mathrm{On}(u)$.

Proof. Given α and $u \in \alpha$,

- $u \subseteq \alpha$ since α is a transitive set,
- therefore u is well-ordered by $\in$; also,
- the set $w = \{v \in \alpha : \ v \in u\}$ is an initial segment of α, so
- since $u \subseteq \alpha$, $w = u$, so u is an initial segment of α, so
- if $v \in u$ and $w \in v$ then $w \in u$, so
- u is a transitive set, and an ordinal. ▮

(3.11) Lemma *Given $\alpha \in \beta \in \gamma$, $\alpha \in \gamma$.*

Proof. Follows from transitivity of γ. ▮

(3.12) Lemma *If u is an initial segment of α then $u \in \alpha$ or $u = \alpha$; in any case, u is an ordinal.*

Proof. By the characterisation of initial segments (Lemma 3.6), if $u \neq \alpha$ then u has the form $\{v \in \alpha \colon v \in t\}$ for some element t of α. In the latter case, $u = \alpha \cap t = t$ since α is transitive. ▮

(3.13) Lemma *Given ordinals α and β, exactly one of the following conditions holds:* (i) $\alpha \in \beta$; (ii) $\alpha = \beta$; (iii) $\beta \in \alpha$.

Proof. Let $\gamma = \alpha \cap \beta$.

Claim that γ is an initial segment of α. Suppose that $v \in \gamma$: then $v \in \alpha$, so v is an ordinal and $v \subseteq \alpha$.

If $u \in v$ then $u \in \alpha$. Similarly, if $v \in \gamma$ then $v \in \beta$ and if $u \in \gamma$ then $u \in \beta$.

Therefore if $u \in \gamma$ and $v \in u$ then $v \in \gamma$. Therefore γ is an initial segment of α, and of β. From the classification of initial segments we get

$$(\gamma = \alpha \vee \gamma \in \beta) \wedge (\gamma \in \alpha \vee \gamma = \beta)$$

This gives four cases of which one is impossible: $\gamma \in \alpha$ and $\gamma \in \beta$ is impossible, because then $\gamma \in \gamma \in \alpha$ and $\{\gamma\}$ would have no minimal element in the well-ordered set α.

The other three cases cover (i), (ii), (iii).

The trichotomy is strict: exercise. ∎

(3.14) Lemma *Every nonempty set of ordinals has a least element.*

Proof. Let y be any nonempty set of ordinals; fix $\beta \in y$. If $\beta \cap y = \varnothing$, then for any $\gamma \in y$, $\gamma \notin \beta$, so β is the least element of y. Otherwise, $\beta \cap y$ has a least element α. Claim that α is the least element of y. Otherwise, there exists some $\gamma \in \alpha \cap y$. But $\alpha \subseteq \beta$ since β is transitive, so $\gamma \in \beta \cap y$, contradicting the choice of α. ∎

(3.15) Lemma *Any transitive set of ordinals is an ordinal.*

Proof. Let x be a transitive set of ordinals. By Lemma 3.13, $(x, \in)$ satisfies the trichotomy, and by Lemma 3.11, $\in$ acts transitively on x. By Lemma 3.14, x satisfies the well-ordering condition. Hence x is an ordinal. ∎

(3.16) Lemma $\alpha \leqslant \beta \iff \alpha \subseteq \beta$.

Proof. If $\alpha \in \beta$ then $\alpha \subseteq \beta$ since β is transitive. It follows easily that $\alpha \leqslant \beta \Rightarrow \alpha \subseteq \beta$.

Conversely, if $\alpha \nleqslant \beta$ then $\beta \in \alpha$ by Lemma 3.13. This implies $\alpha \nsubseteq \beta$, since otherwise $\beta \in \beta$, contradicting that Lemma. ∎

(3.17) Notation On *denotes the 'collection' of ordinals.*

(3.18) Lemma *The collection* On *is not a set.*

Proof. Otherwise, by Lemmas 3.10 and 3.15, On would be an ordinal α, and α would be a member of α, which is impossible. ∎

(3.19) Definition *Given an ordinal α, one writes $\alpha+1$ for $\alpha \cup \{\alpha\}$, and calls it the* SUCCESSOR *of α.*

(3.20) Lemma *$\alpha + 1$ is the least ordinal greater than α.*

Proof. Let $\beta = \alpha + 1 = \alpha \cup \{\alpha\}$. If $w \in \beta$ then either $w \in \alpha$, so $w \subseteq \alpha \subseteq \beta$, or $w = \alpha$, so $w \subseteq \beta$ by definition of β. Thus β is transitive, and therefore an ordinal (Lemma 3.15).

Let γ be any ordinal greater than α. By definition, $\alpha \in \gamma$, and therefore $\alpha \subseteq \gamma$ since γ is transitive. Therefore $\beta \subseteq \gamma$, so $\beta \leqslant \gamma$ by Lemma 3.16. ▮

The 'natural numbers.' The empty set $\varnothing$ is an ordinal and it is clearly the smallest ordinal, which can be identified with the natural number 0. Its successor is $\{\varnothing\}$ which can be identified with 1; its successor is $\{\varnothing, \{\varnothing\}\}$, which can be identified with 2, and so on. All the natural numbers can be identified with certain ordinals, just as in arithmetic theories the natural numbers n could be identified with 'numerals' $\bar{n}$.

(3.22) Lemma *If x is any set then the ordinals in x have a least upper bound α (default $\varnothing$). We write $\alpha = \sup(x)$.*

Proof. Let $w = \bigcup\{\beta\colon \beta \in x\}$. If $\gamma \in \delta \in w$, then for some $\beta \in x$, $\delta \in \beta$; then $\gamma \in \beta$, so $\gamma \in w$. Thus w, being a transitive set of ordinals, is an ordinal, so let us relabel it α. If $\beta \in x$ then $\beta \leqslant \alpha$ by definition (i.e., $\beta \subseteq \alpha$). Conversely, if $\gamma < \alpha$ then for some $\beta \in x$, $\gamma \in \beta$. Therefore α is the least upper bound of all ordinals in x. ▮

(3.23) Lemma Principle of transfinite induction for ordinals.

- *If $\mathcal{F}(\alpha)$ is a formula of* ZF*, such that the following can be proved in* ZF*:*

  for every α, if $\mathcal{F}(\beta)$ holds for every $\beta \in \alpha$, then $\mathcal{F}(\alpha)$ follows,

- *then $\mathcal{F}(\alpha)$ is true for every α.*

(In other words, the formula $\forall\alpha\mathcal{F}(\alpha)$ can be proved in ZF*.)*

Proof. If $\neg\forall\alpha\mathcal{F}(\alpha)$, fix a θ such that $\neg\mathcal{F}(\theta)$, and let $u = \{v \in \theta + 1\colon \neg\mathcal{F}(v)\}$. By the comprehension principle, u is a well-defined set.

Indeed, u is a nonempty set of ordinals; let α be its least element. If $\beta \in \alpha$, then $\beta \in \theta + 1$ since $\beta \in \alpha \in u$, and $\beta \notin u$ since α is minimal. Hence $\mathcal{F}(\beta)$. Therefore $(\forall\beta \in \alpha)\mathcal{F}(\beta)$. Therefore $\mathcal{F}(\alpha)$, a contradiction. ▮

(3.24) Definition *An* order isomorphism *between well-ordered sets $(x, <)$ and $(x', <')$ is a bijection $f\colon x \to x'$ such that $u < v \in x \Rightarrow f(u) <' f(v)$.*

(3.25) Theorem *For every well-ordered set $(x, <)$ there exists a unique order-isomorphism f from a unique ordinal onto x.*

Proof. This part has been suppressed. The fact is important and interesting, but the proof (using transfinite induction) is rather long, and it will be omitted or deferred until later.