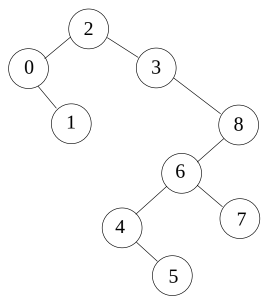
Maths 3467 quizzes
1: Thursday 3/10/13 answers

(1) The following is an inorder *labelling* with preorder *sequencing* of the nodes in a binary tree. Construct the tree.

$$2\ 0\ 1\ 3\ 8\ 6\ 4\ 5\ 7$$

Answer.



(2) Show that *any* binary tree can be constructed as a search tree from *some* key sequence.

Answer. One construction is to associate the *inorder* rank as the key of every node ($1 \ldots n$ or $0 \ldots n-1$: it makes no difference), and insert them in *preorder sequence.*

Outline sketch proof (tentative). One could establish the following by some kind of inductive argument: *Suppose that B is a nonempty binary search tree with left subtree B_1, key M stored at the root, and right subtree B_2. Suppose that k is a key different from M. Then the result of inserting k into the tree B is the same as if we inserted k into B_1 if $k < M$ or into B_2 if $k > M$.*

Given T, let

$$r_1\, r_2\ \ldots\ r_n$$

be the inorder ranks of nodes presented in preorder, so r_1 is the key M discussed above and in fact the sequence can be written as

$$M\ \ell_1 \ldots \ell_i\ \ h_1 \ldots h_j$$

where $M = i + 1$, $\ell_1 \ldots \ell_i$ presents the inorder ranks in T_1 in preorder, and $h_1 \ldots h_j$ presents the inorder ranks in T_2 in preorder.

Using induction on n (cases $n = 0, 1$ are trivial), we can assume that $\ell_1 \ldots \ell_i$ on its own would construct a tree B_1 isomorphic to T_1, and $h_1 \ldots h_j$ would construct B_2 isomorphic to T_2; by the above remarks the full key sequence would produce a tree B isomorphic to T. ∎

(3) (1) suggests that a binary tree is uniquely defined given its nodes presented in preorder and in inorder (the two sequences yield the structure of the tree). Show that the preorder and postorder sequences together are not enough to fix the tree uniquely (exhibit two very small trees which agree in preorder and in postorder).

Different trees with the same preorder ba and postorder ab.

(4) Use Stirling's approximation

$$k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$$

to give an approximation of the Catalan numbers.

Answer.

$$\frac{\sqrt{4n\pi}\left(\frac{2n}{e}\right)^{2n}}{(n+1)2n\pi\left(\frac{n}{e}\right)^{2n}} = \frac{4^n}{(n+1)\sqrt{n\pi}}$$

(5) Use Stirling's approximation to give an approximation to the average IPL of binary trees (as opposed to binary search trees).

Answer. Writing C_n for the Catalan numbers,

$$\frac{4^n}{C_n} - 3n - 1 =$$
$$\frac{4^n}{\frac{4^n}{(n+1)\sqrt{n\pi}}} - 3n - 1 =$$
$$(n+1)\sqrt{n\pi} - 3n - 1$$

This is $O(n\sqrt{n})$ and $\Omega(n\sqrt{n})$ simultaneously.