There might be a couple of errors in these answers — if you don't agree with an answer, please email `odunlain@maths.tcd.ie`.

HW 01 Due 16/10/06

$$(1)\begin{bmatrix}0\\0\end{bmatrix}(2)\begin{bmatrix}0&1\\1&0\end{bmatrix}(3)\begin{bmatrix}0&1\\0&0\end{bmatrix}(4)\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}(5)\begin{bmatrix}1&0&1&0\\0&0&1&0\\0&0&0&0\end{bmatrix}(6)\begin{bmatrix}1&2&0&0&3\\0&0&0&1&0\\0&0&1&0&1\end{bmatrix}$$

$$(7)\begin{bmatrix}0&1&1\\0&0&0\\0&2&1\end{bmatrix}(8)\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}(9)\begin{bmatrix}0&1&1&0&2\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}(10)\begin{bmatrix}0&1&1&-2&0\\0&0&0&1&1\\0&0&0&0&0\end{bmatrix}$$

Question 1. For the above 10 matrices, list their leading columns, and say whether they are in RREF (20 marks).

	Leading columns	In RREF?
1	none	yes
2	1,2	no
3	2	yes
4	1,3	yes
5	1,3	no

	Leading columns	In RREF?
6	1,3,4	no
7	2	no
8	1,2	yes
8	2,4	yes
10	2,4	no

Question 2. Form the augmented matrix for the following system of equations, bring to RREF using annotated Gauss-Jordan elimination **(no other method)**, and hence solve (10 marks).

$$x - y + 4z = 9, \quad 3x - y + 10z = 25, \quad x + 2z = 6$$

```
1 -1  4  9   =R1          1  0  3  8   -3*R3
3 -1 10 25   -3*R1        0  1 -1 -1   +1*R3
1  0  2  6   -1*R1        0  0 -1 -2  *(-1)  =R3

1 -1  4  9   +1*R2        1  0  0  2
0  2 -2 -2 *(1/2) =R2     0  1  0  1
0  1 -2 -3   -1*R2        0  0  1  2  in rref
```

Solution $x = 2, y = 1, z = 2$.

Question 3. Form the augmented matrix for the following system of equations, bring to RREF using annotated Gauss-Jordan elimination **(no other method)**, and hence solve (10 marks).

$$-w-2x+2y+2z = 3, \quad 3w+6x-6y-5z = -8, \quad -2w-4x+6y+4z = 8, \quad -2w-4x+5y+7z = 11$$

```
 -2 -4  6  4  8   +2*R1      0  0  0  1  1   =R3
 -2 -4  5  7 11   +2*R1      0  0  0  3  4   -3*R3

  1  2 -2 -2 -3              1  2  0  0  1   -1*R4
  0  0  0  1  1  swap        0  0  1  0  1   -1*R4
  0  0  2  0  2  swap        0  0  0  1  1   -1*R4
  0  0  1  3  5              0  0  0  0  1   =R4

  1  2 -2 -2 -3  +2*R2       1  2  0  0  1
  0  0  2  0  2  *(1/2) =R2  0  0  1  0  1
  0  0  0  1  1              0  0  0  1  1
  0  0  1  3  5  -1*R2       0  0  0  0  1  in rref
```

No solution.

Question 4. Form the augmented matrix for the following system of equations, bring to RREF using annotated Gauss-Jordan elimination **(no other method)**, and hence solve (10 marks).

$$-w-2x+2y+2z=3,\quad 3w+6x-6y-5z=-8,\quad -2w-4x+6y+4z=8,\quad -2w-4x+5y+7z=10$$

```
 -1 -2  2  2  3 *(-1) =R1    1  2  0 -2 -1   +2*R3
  3  6 -6 -5 -8 -3*R1        0  0  1  0  1
 -2 -4  6  4  8 +2*R1        0  0  0  1  1   =R3
 -2 -4  5  7 10 +2*R1        0  0  0  3  3   -3*R3

  1  2 -2 -2 -3              1  2  0  0  1
  0  0  0  1  1  swap        0  0  1  0  1
  0  0  2  0  2  swap        0  0  0  1  1
  0  0  1  3  4              0  0  0  0  0  in rref

  1  2 -2 -2 -3 +2*R2
  0  0  2  0  2 *(1/2) =R2
  0  0  0  1  1
  0  0  1  3  4 -1*R2
```

Solutions $w=1-2r, x=r, y=1-r, z=1$, where r is arbitrary.

HW 2 Due TUESDAY 24/10/06

(1) (18 marks) Let

$$A=\begin{bmatrix}3&2\\4&1\end{bmatrix},\quad B=\begin{bmatrix}2&3\\3&5\end{bmatrix},\quad C=\begin{bmatrix}5&-3\\-3&2\end{bmatrix}.$$

$$(\text{i})AB = \begin{bmatrix} 12 & 19 \\ 11 & 17 \end{bmatrix}, \quad (\text{ii})BA = \begin{bmatrix} 18 & 7 \\ 29 & 11 \end{bmatrix}, \quad (\text{iii})BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(\text{iv})CB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\text{v})A(BC) = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}, \quad (\text{vi})(AB)C = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}.$$

(2) (8 marks) Invert the matrix

$$\begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix}$$

(i) Using the $1/(ad - bc)\ldots$ formula

$$\frac{1}{7-10}\begin{bmatrix} 7 & -5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-7}{3} & \frac{5}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{bmatrix}$$

(ii) Using annotated GJE on the usual 2×4 matrix.

```
1    5    1    0   =R1              1    5    1    0   -5*R2
2    7    0    1   -2*R1            0   -3   -2    1  *(-1/3)   =R2

1    0 -7/3  5/3
0    1  2/3 -1/3  in rref
```

(3) (10 marks). Using annotated GJE on the usual 3×6 matrix, determine whether the following matrix is invertible, and calculate the inverse if it is. (There are some fractions in the calculation.)

$$\begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 2 \\ 3 & 2 & 17 \end{bmatrix}$$

```
1  1  6  1  0  0 =R1              1  0  5  1 -1/2  0 -1*R3
0  2  2  0  1  0                  0  1  1  0  1/2  0
3  2 17  0  0  1 -3*R1            0  0  0 -3  1/2  1  *(-1/3) =R3
                                  ... NOT INVERTIBLE! Can stop here.

1  1  6  1  0  0 -1*R2            1  0  5  0 -1/3  1/3
0  2  2  0  1  0  *(1/2)=R2       0  1  1  0  1/2  0
0 -1 -1 -3  0  1 +1*R2            0  0  0  1 -1/6 -1/3  in rref
```

(4) (14 marks). Do the same with the following 4×4 matrix. (There should be only integers in the calculation.)

$$\begin{bmatrix} 1 & -3 & 1 & -6 \\ 1 & -2 & 1 & -5 \\ 3 & -9 & 3 & -19 \\ 0 & -2 & 1 & -2 \end{bmatrix}$$

```
3 -9  3 -19  0  0  1  0  -3*R1        0  0  1  0 -2  2  0  1  =R3
0 -2  1 -2   0  0  0  1               0  0  0 -1 -3  0  1  0

1 -3  1 -6  1  0  0  0  +3*R2         1  0  0 -3  0  1  0 -1  +3*R4
0  1  0  1 -1  1  0  0  =R2           0  1  0  1 -1  1  0  0  -1*R4
0  0  0 -1 -3  0  1  0                0  0  1  0 -2  2  0  1
0 -2  1 -2  0  0  0  1  +2*R2         0  0  0 -1 -3  0  1  0 *(-1) =R4

1  0  1 -3 -2  3  0  0                1  0  0  0  9  1 -3 -1
0  1  0  1 -1  1  0  0                0  1  0  0 -4  1  1  0
0  0  0 -1 -3  0  1  0 swap           0  0  1  0 -2  2  0  1
0  0  1  0 -2  2  0  1 swap           0  0  0  1  3  0 -1  0 in rref
                                                  <-inverse->
```

HW 3 Due 31/10/06

(1) (12 marks) Let $P = (2,3), Q = (4,5)$. (i) $P + Q = (6,8)$ (ii) $P - Q = (-2,-2)$ (iii) $\vec{PQ} = (2,2)$ (iv) $\vec{QP} = (-2,-2)$.

(2) (12 marks) Let $P = (2,3,4)$, $Q = (5,6,7)$. Calculate (i) $P + Q = (7,9,11)$ (ii) $P - Q = (-3,-3,-3)$ (iii) $\vec{PQ} = (3,3,3)$ (iv) $\vec{QP} = (-3,-3,-3)$.

(3) (13 marks) Calculate the intersection of the three planes

$$2x + 4z = 10, \quad 2x + 2y + 10z = 26, \quad 3x + y + 8z = 21.$$

```
2  0  4 10  *(1/2)  =R1        1  0  2  5   -2*R3
2  2 10 26   -2*R1             0  1  3  8   -3*R3
3  1  8 21   -3*R1             0  0 -1 -2  *(-1)  =R3

1  0  2  5                     1  0  0  1
0  2  6 16  *(1/2)  =R2        0  1  0  2
0  1  2  6   -1*R2             0  0  1  2  in rref
```

Solution: $x = 1, y = 2, z = 2$.

(4) (13 marks) Calculate the intersection of the three planes

$$2x + 2y - 2z = -2, \quad x + y + z = 3, \quad x + y - 4z = -7$$

```
2  2 -2 -2  *(1/2) =R1          1  1  0  1
1  1  1  3   -1*R1              0  0  1  2
1  1 -4 -7   -1*R1              0  0  0  0  in rref

1  1 -1 -1   +1*R2
0  0  2  4   *(1/2) =R2
0  0 -3 -6   +3*R2
```

HW 4 Due 7/11/06

(1) (12 marks). Let $P = (2, 5)$, $Q = (3, 4)$. (i) Calculate the matrix for converting from 'oblique' coordinates (in the ordered basis (P, Q)) to cartesian coordinates. (ii) Calculate the matrix for converting from cartesian to P, Q-coordinates

$$\text{(i)}S = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}, \quad \text{(ii)}S^{-1} = \frac{1}{8-15}\begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-4}{7} & \frac{3}{7} \\ \frac{5}{7} & \frac{-2}{7} \end{bmatrix}.$$

(iii) Calculate the 'new' coordinates of $(1, 2)$. $(2/7, 1/7)$.

(iv) Calculate the cartesian coordinates of the point whose new coordinates are $(1, 2)$. $(8, 13)$.

(2)(13 marks). Calculate the (i) new-to-old and (ii) old-to-new coordinate change matrices for the new ordered basis $P = (1, -1, 1), Q = (2, 2, -1), R = (-5, -6, 2)$.

$$\text{(i)}S = \begin{bmatrix} 1 & 2 & -5 \\ -1 & 2 & -6 \\ 1 & -1 & 2 \end{bmatrix} \text{(ii) see below} S^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{-7}{5} & \frac{-11}{5} \\ \frac{1}{5} & \frac{-3}{5} & \frac{-4}{5} \end{bmatrix}$$

```
  1     2    -5     1     0     0    =R1
 -1     2    -6     0     1     0    +1*R1
  1    -1     2     0     0     1    -1*R1

  1     2    -5     1     0     0    -2*R2
  0     4   -11     1     1     0   *(1/4)   =R2
  0    -3     7    -1     0     1    +3*R2

  1     0   1/2   1/2  -1/2     0    -1/2*R3
  0     1 -11/4   1/4   1/4     0    +11/4*R3
  0     0  -5/4  -1/4   3/4     1   *(-4/5)   =R3

  1     0     0   2/5  -1/5   2/5
  0     1     0   4/5  -7/5 -11/5
  0     0     1   1/5  -3/5  -4/5   in rref
```

(3) (12 marks). (i) Give a point on the line $x + 2y = 3$. $(1, 1)$.

(ii) Give a point P such that $\vec{OP}$ is perpendicular to this line. $(1, 2)$.

(iii) Calculate the positive normal to the vector $\vec{OP}$ where $P = (15, -13)$. $(13, 15)$.

(iv) Calculate the distance from $(2, 3, 1)$ to $(4, 1, 2)$. $\sqrt{(4-2)^2 + (1-3)^2 + (2-1)^2} = \sqrt{9} = 3$.

(4) (13 marks). Apply the Gram-Schmidt orthogonalisation procedure to the points $(1, 2, 3)$ and $(4, 5, 6)$. (Deferred to HW 5.)

HW 5 Due 7/11/06

of a "positive normal" to $\vec{PQ}$.

$\vec{PQ} \equiv (-1,1)$: positive normal $(-1,-1)$;

$$(-1,-1)\cdot \vec{PX} = 0$$

$$(-1,-1)\cdot \vec{OX} = (-1,-1)\cdot \vec{OP} :$$

$$-x - y = -5.$$

(2)(10 marks) Find a point on the line $3x+4y=5$ and hence interpret the equation in the form $\{X:\ \vec{ON}\cdot\vec{PX}=0\}$.

Take, for example, $x=0$ and we get $y=5/4$. The normal direction is $(3,4)$. This gives

$$\{X : (3,4)\cdot \vec{PX} = 0\},$$

where $P=(0,5/4)$. Alternatively,

$$\{(x,y) : (3,4)\cdot(x, y-5/4) = 0\}.$$

(3)(10 marks) Give an equation for the plane through the points $(1,-1,1)$, $(2,2,-1)$, and $(-5,-6,2)$, using a cross product.

Take $P=(1,-1,1)$, $Q=(2,2,-1)$, and $R=(-5,-6,2)$. Then $\vec{PQ}$ is equivalent to $Q-P=(1,3,-2)$ and $\vec{PR}$ is equivalent to $R-P=(-6,-5,1)$. Their cross product is

$$\left(\begin{vmatrix} 3 & -2 \\ -5 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & -2 \\ -6 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ -6 & -5 \end{vmatrix} \right) = (-7,11,13).$$

The equation can be written

$$(-7,11,13)\cdot\vec{OX} = (-7,11,13)\cdot\vec{OP},$$

or

$$-7x + 11y + 13z = -5.$$

(4) (20 marks). Apply the Gram-Schmidt orthogonalisation procedure to the points $(1,2,3)$, $(4,5,6)$, and $(0,0,1)$. (The third point is almost irrelevant).

$W_1 = P_1 = (1,2,3)$;

$$U_1 = (1,2,3)/\sqrt{14}.$$

$$W_2 = P_2 - \frac{\vec{OP_2}\cdot\vec{OW_1}}{\vec{OW_1}\cdot\vec{OW_1}} W_1 = (4,5,6) - \frac{32}{14}(1,2,3) = (24,6,-12)/14$$

$$U_2 = (4,1,-2)/\sqrt{21}.$$

$\vec{OW_3} = \vec{OP_1}\times\vec{OP_2} = (-3,6,-3)$:

$$U_3 = (-1,2,-1)/\sqrt{6}.$$

This U_3 may differ in sign from the one provided by Gram-Schmidt orthogonalisation, but that is not vitally important.

$$\begin{bmatrix} m \\ c \end{bmatrix} = (AA^T)^{-1}A^Tb,$$

where

$$A = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}^T, \quad \text{and} \quad b = [1\ 2\ 0\ 3]^T.$$

$$\begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 56 & 12 \\ 12 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 22 \\ 6 \end{bmatrix}.$$

Solution: $m = 1/5, c = 9/10$.

(2)(12 marks) Which of the following maps from E^2 to E^2 are linear? If not, why not?. (i) $(x, y) \mapsto (x + 1, y - 1)$ (ii) $(x, y) \mapsto (x^2, x^2)$ (iii) $(x, y) \mapsto (x + y, x - y)$

Answers. (i) No, because O is not mapped to O.

(ii) No, since $(1, 1) \mapsto (1, 1)$ but $2(1, 1) \mapsto (4, 4)$.

(iii) Yes.

(3)(10 marks) Construct the matrix of the map which projects points orthogonally onto the line $x + 2y = 0$.

Answer. Choose a point P on the line, such as $(2, -1)$. Let Q be the positive normal to P, i.e., $Q = (1, 2)$. With respect to the basis P, Q the 'new' matrix A' is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The 'change of basis' matrix is

$$S = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}; \quad S^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

$$A = SA'S^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{5} & \frac{-1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$$

(4)(18 marks) Let $P = (1, -3, -1)$, $Q = (1, -2, 1)$, $R = (-4, 10, 1)$. Calculate the matrix of the linear map whose matrix with respect to the basis P, Q, R is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Answer. The change-of-basis matrix is

$$S = \begin{bmatrix} 1 & 1 & -4 \\ -3 & -2 & 10 \\ -1 & 1 & 1 \end{bmatrix}$$

Invert using GJE:

```
 -1  1  1  0  0  1 +1*R1

  1  1 -4  1  0  0 -1*R2
  0  1 -2  3  1  0 =R2
  0  2 -3  1  0  1 -2*R2

  1  0 -2 -2 -1  0 +2*R3
  0  1 -2  3  1  0 +2*R3
  0  0  1 -5 -2  1 =R3
RREF:
  1  0  0 -12 -5  2
  0  1  0  -7 -3  2
  0  0  1  -5 -2  1
```

$A = SA'S^{-1} =$

$$\begin{bmatrix} 1 & 1 & -4 \\ -3 & -2 & 10 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -12 & -5 & 2 \\ -7 & -3 & 2 \\ -5 & -2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 4 & -4 \\ -3 & -10 & 11 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -12 & -5 & 2 \\ -7 & -3 & 2 \\ -5 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -20 & -9 & 6 \\ 51 & 23 & -15 \\ -8 & -3 & 2 \end{bmatrix}$$

The order of computation was $(SA')S^{-1}$. Alternatively, one could have computed $S(A'S^{-1}) =$

$$\begin{bmatrix} 1 & 1 & -4 \\ -3 & -2 & 10 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -31 & -13 & 7 \\ -29 & -12 & 7 \\ -10 & -4 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -9 & 6 \\ 51 & 23 & -15 \\ -8 & -3 & 2 \end{bmatrix}$$

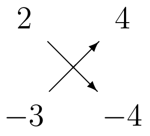
HW 7 Due 28/11/06

(1)(12 marks) Calculate the determinants of the following matrices by reducing to upper triangular form.

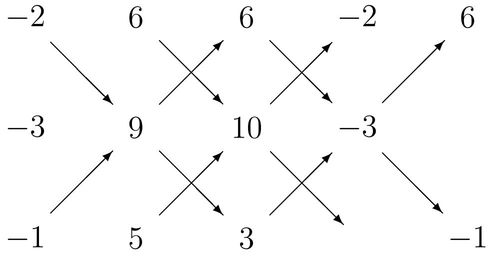
```
(a)
  2  4 = R1                   2 4            Determinant is 4.
 -3 -4 + (3/2)*R1             0 2

(b)
 -2  6  6  = R1                   -2  6  6  UTF, det = -4
 -3  9 10  -(3/2)*R1               0  2  0
 -1  5  3  -(1/2)*R1               0  0  1

 -2  6  6                         One swap: determinant is 4.
  0  0  1  swap
  0  2  0  swap
```



$(2)(-4)-(-3)(4)=4.$



$(-2)(9)(3)+(6)(10)(-1)+(6)(-3)(5)-(-1)(9)(6)-(5)(10)(-2)-(-1)(9)(6)=$
$-54-60-90+54+100+54=4.$

(3)(14 marks) Calculate the following determinant by reducing to upper triangular form.

$$\begin{vmatrix} 2 & 0 & 0 & 6 \\ -1 & -2 & 2 & -5 \\ 2 & -3 & 3 & 4 \\ 3 & 0 & -2 & 7 \end{vmatrix}$$

```
  2  0  0  6
 -1 -2  2 -5
  2 -3  3  4
  3  0 -2  7

inspect column no. 1
subtract -1/2* row 1 from row 2
subtract 1* row 1 from row 3
subtract 3/2* row 1 from row 4

  2  0  0  6
  0 -2  2 -2
  0 -3  3 -2
  0  0 -2 -2
```

```
inspect column no. 2
subtract 3/2* row 2 from row 3

2  0  0  6
0 -2  2 -2
0  0  0  1
0  0 -2 -2

inspect column no. 3
swap rows 3, 4

2  0  0  6
0 -2  2 -2
0  0 -2 -2
0  0  0  1
determinant is -8
```

(4)(12 marks) Calculate the same determinant using cofactor expansion along the first row.

$$2\begin{vmatrix} -2 & 2 & -5 \\ -3 & 3 & 4 \\ 0 & -2 & 7 \end{vmatrix} - 6\begin{vmatrix} -1 & -2 & 2 \\ 2 & -3 & 3 \\ 3 & 0 & -2 \end{vmatrix}$$

$$=2(-42+0-30-0-16+42)-6(-6-18+0+18+0-8)=$$
$$2(-46)-6(-14)=-92+84=-8$$

(1) Invert the following matrix using (i) the Adjoint Matrix method and (ii) Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -3 & 1 \\ -1 & 5 & -3 \\ -3 & 12 & -5 \end{bmatrix} \quad \text{(i) } P = (1, -3, 1), Q = (-1, 5, -3), R = (-3, 12, -5)$$

```
OQxOR: -1  5 -3  ORxOP:  -3 12 -5  OPxOQ:  1 -3  1
       -3 12 -5           1 -3  1         -1  5 -3
       --------          --------         --------
       11  4  3          -3 -2 -3          4  2  2
Determinant = OP . (OQxOR) = 2.
```

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{2}\begin{bmatrix} 11 & -3 & 4 \\ 4 & -2 & 2 \\ 3 & -3 & 2 \end{bmatrix}.$$

```
(ii) By GJE.
 1  -3   1    1    0    0 =R1        1   0   -2   5/2   3/2    0 +2*R3
-1   5  -3    0    1    0 +1*R1      0   1   -1   1/2   1/2    0 +1*R3
-3  12  -5    0    0    1 +3*R1      0   0    1   3/2  -3/2    1 =R3

 1  -3   1    1    0    0 +3*R2      1   0    0 11/2  -3/2    2
 0   2  -2    1    1    0 *(1/2)=R2  0   1    0    2    -1    1
 0   3  -2    3    0    1 -3*R2      0   0    1   3/2  -3/2    1 in rref
```

Answer same as in (i).

(2) Test the following lists of points for linear independence.

(i) $(1, -2, 0), (3, -7, -1), (2, -7, -3)$

(ii) $(1, 3, 2), (-2, -7, -7), (0, -1, -3)$

(iii) $(-1, -3, 3), (2, 7, -8), (0, -2, 2)$.

```
  1  3  2   =R1    1  3  2    -3*R2      1  0 -7
 -2 -7 -7 +2*R1    0 -1 -3 *(-1)=R2      0  1  3
  0 -1 -3          0 -1 -3    +1*R2      0  0  0  in rref

  1 -2  0   =R1    1 -2  0   +2*R2       1  0  2
  3 -7 -1 -3*R1    0 -1 -1*(-1)=R2       0  1  1
  2 -7 -3 -2*R1    0 -3 -3   +3*R2       0  0  0  in rref

 -1  2  0 *(-1)=R1        1  0 -4   +4*R3
 -3  7 -2    +3*R1        0  1 -2   +2*R3
  3 -8  2    -3*R1        0  0 -2  *(-1/2)  =R3

  1 -2  0 +2*R2           1  0  0
```

Therefore (i) and (ii) are linearly dependent and (iii) is linearly independent.

Alternatively we can calculate the determinants

```
        0  7  18
 1  3  2  1  3
-2 -7 -7 -2         ... 21 + 4 - 7 - 18 = 0: linearly dependent
 0 -1 -3  0 -1
        21 0  4

        0  7  18
 1 -2  0  1 -2
 3 -7 -1  3         ... 21 + 4 - 7 - 18 = 0: linearly dependent
 2 -7 -3  2 -7
        21  4  0

        0  -16 -12
-1  2  0 -1  2
-3  7 -2 -3         ... 14 - 12  + 16  + 12 = 30: linearly independent
 3 -8  2  3 -8
        14 -12 0
```

(3) Test the columns of the following matrix for linear independence.

$$\begin{bmatrix} -1 & -2 & 3 & 4 \\ 3 & 8 & -9 & -18 \\ 1 & 0 & -3 & 3 \\ 1 & 1 & -4 & 0 \end{bmatrix}$$

```
-1 -2  3   4 *(-1)=R1     1  2 -3  -4      -2*R2     1  0 -3   2
 3  8 -9 -18     -3*R1     0  2  0  -6 *(1/2)=R2     0  1  0  -3
 1  0 -3   3     -1*R1     0 -2  0   7      +2*R2     0  0  0   1 swap
 1  1 -4   0     -1*R1     0 -1 -1   4      +1*R2     0  0 -1   1 swap

 1  0 -3   2     +3*R3     1  0  0  -1 +1*R4     1  0  0   0
 0  1  0  -3               0  1  0  -3 +3*R4     0  1  0   0
 0  0 -1   1 *(-1)=R3     0  0  1  -1 +1*R4     0  0  1   0
 0  0  0   1               0  0  0   1   =R4     0  0  0   1 in rref
```

Therefore they are linearly independent.

(4) In the following matrix, the third column depends on the others. Use the RREF to express the third column as a linear combination of the others.

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ -2 & -6 & -20 & 14 \\ -3 & -7 & -24 & 14 \end{bmatrix}.$$

```
 (1)                        (2)                             (3)
  0   1   3  -2  swap  -2  -6 -20  14 *(-1/2)=R1   1   3  10  -7
  0   0   0  -2         0   0   0  -2              0   0   0  -2  swap
 -2  -6 -20  14  swap   0   1   3  -2              0   1   3  -2  swap
 -3  -7 -24  14        -3  -7 -24  14 +3*R1        0   2   6  -7

 (4)                        (5)                             (6)
  1   3  10  -7 -3*R2   1   0   1  -1       +1*R3  1   0   1   0
  0   1   3  -2   =R2   0   1   3  -2       +2*R3  0   1   3   0
  0   0   0  -2         0   0   0  -2 *(-1/2)=R3   0   0   0   1
  0   2   6  -7 -2*R2   0   0   0  -3       +3*R3  0   0   0   0  in rref
```

As we see from the RREF, the third column A'_3 of the RREF is non-leading and depends on the first two: $A'_3 = A'_1 + 3A'_2$. In the original matrix, $A_3 = A_1 + 3A_2$. (Equivalently, $A_3 = A_1 + 3A_2 + 0A_4$.)