

(1) Solve $D^2y - 2Dy - 3y = e^x$

$$(D - 3)(D + 1)y = e^x$$

$$e^{3x} D e^{-3x} e^{-x} D e^x y = e^x$$

$$D e^{-4x} D e^x y = e^{-2x}$$

$$e^{-4x} D e^x y = -\frac{e^{-2x}}{2} \text{ (constant doesn't matter)}$$

$$D e^x y = -\frac{e^{2x}}{2}$$

$$e^x y = -\frac{e^{2x}}{4}$$

$$y = -\frac{e^x}{4} \text{ particular solution}$$

$$y = A e^{3x} + B e^{-x} - \frac{e^x}{4}$$

(2) Solve $D^2y - 2Dy - 3y = e^{-x}$

$$(D - 3)(D + 1)y = e^{-x}$$

$$e^{3x} D e^{-3x} e^{-x} D e^x y = e^{-x}$$

$$D e^{-3x} e^{-x} D e^x y = e^{-4x}$$

$$e^{-3x} e^{-x} D e^x y = -\frac{e^{-4x}}{4}$$

$$D e^x y = -\frac{1}{4}$$

$$e^x y = -\frac{x}{4}$$

$$y = -\frac{x e^{-x}}{4} \text{ particular}$$

$$y = A e^{3x} + B e^{-x} - \frac{x e^{-x}}{4} \text{ general}$$

(3) Solve $D^2y - 2Dy + y = 1$

$$(D - 1)(D - 1)y = 1$$

$$e^x De^{-x} e^x De^{-x} y = 1$$

$$De^{-x} e^x De^{-x} y = e^{-x}$$

$$e^{-x} e^x De^{-x} y = -e^{-x}$$

$$De^{-x} y = -e^{-x}$$

$$e^{-x} y = e^{-x}$$

$$y = 1 \text{ (particular)}$$

$$y = Ae^x + Bxe^x + 1 \text{ (general)}$$

(4) Solve $D^2y - 2Dy + 2y = 1$ (general real solution)

$\lambda^2 - 2\lambda + 2$ has roots α, β where $\alpha = 1 + i$ and $\beta = 1 - i$.

$$e^{\alpha x} De^{-\alpha x} e^{\beta x} De^{-\beta x} y = 1$$

$$De^{-\alpha x} e^{\beta x} De^{-\beta x} y = e^{-\alpha x}$$

$$e^{-\alpha x} e^{\beta x} De^{-\beta x} y = -\frac{e^{-\alpha x}}{\alpha}$$

$$De^{-\beta x} y = -\frac{e^{-\beta x}}{\alpha}$$

$$e^{-\beta x} y = \frac{e^{-\beta x}}{\alpha\beta} = \frac{e^{-\beta x}}{2}$$

$$y = \frac{1}{2} \text{ (particular)}$$

$$y = Ae^{\alpha x} + Be^{\beta x} + \frac{1}{2} \text{ (general complex)}$$

$$y = Ae^x \cos x + Be^x \sin x + \frac{1}{2} \text{ (general real solution)}$$