Mathematics 1E2 2006–07 Hilary term answers

Homework 09 due 15/1/07

This homework is due IN CLASS on MONDAY. ***Pay close attention to the regulations about homeworks and attendance, which will be in force for the next two terms.***

(1) (18 marks). Let e_1, e_2, e_3 be the following EROs on matrices of height 3:
e_1: scale row 3 by 3; e_2: swap rows 1 and 3; e_3: Subtract 2×row 3 from row 2.
Give the elementary matrices E_1, E_2, E_3, for these operations, and their inverses.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$E_2^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \ E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2) (10 marks) Calculate $P = E_3E_2E_1$. Hint: direct multiplication is not the easiest way.

```
Start with E_1
      1  0  0                0  0  3              0  0  3
      0  1  0                0  1  0             -2  1  0
      0  0  3  Apply e_2     1  0  0  then e_3    1  0  0 = P
```

(3) (10 marks) Calculate $E_1^{-1}E_2^{-1}E_3^{-1}$. Again, there are easier ways of doing it.

```
Start with E_3 inverse
     1  0  0  e_2 inverse  0  0  1 e_1 inverse  0   0  1  ...= P inverse
     0  1  2               0  1  2              0   1  2
     0  0  1               1  0  0             1/3  0  0
```

(4) (12 marks) Let $A = [a_{ij}]_{\ell\times m}$, $B = [b_{jk}]_{m\times n}$, $AB = [c_{ik}]$. Note that B^TA^T is well-defined. Let $A^T = [e_{ji}]$, $B^T = [d_{kj}]$, and $B^TA^T = [f_{ki}]$. With this notation, show that $B^TA^T = (AB)^T$.

Proof. We need to show that for $1 \le k \le n$, $1 \le i \le \ell$,

$$f_{ki} = c_{ik}.$$

$$f_{ki} = \sum_j d_{kj}e_{ji} = \sum_j b_{jk}a_{ij} = \sum_j a_{ij}b_{jk} = c_{ik},$$

as required. **Q.E.D.**

Homework 10 due 22/1/07

(1) Answer true or false:
(i) An elementary matrix must have nonzero determinant. **TRUE**

(ii) If $(AB)^{-1}$ exists, A^{-1} exists. **FALSE**

(iii) $\begin{vmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{vmatrix} = cef$. **FALSE**

(2) Answer true or false:
(i) The product of two upper triangular matrices is **lower** triangular. **FALSE**
(ii) One can **not** have three linearly independent points in two dimensions. **TRUE**
(iii) Say A is 2×2. Then $\det(2A) = 2\det(A)$ **FALSE**

(3) Assuming B, S are square and compatible and S^{-1} exists, (i) prove $\det(S^{-1}) = 1/\det(S)$

Proof. $\det(S)\det(S^{-1} = det(SS^{-1}) = \det(I) = 1$, so $\det(S^{-1} = 1/\det(S)$. **Q.E.D.**

(ii) $\det(S^{-1}BS) = \det(B)$.

Proof.

$$\det(S^{-1}BS) = \det(S^{-1})\det(B)\det(S) = (1/\det(S))\det(B)\det(S) = \det(B).$$

Q.E.D.

(4) Deduce that $\det(\lambda I - S^{-1}AS) = \det(\lambda I - A)$. (Note: $S^{-1}(\lambda I)S = \lambda I$.)

Proof.

$$\det(\lambda I - S^{-1}AS) = \det(S^{-1}\lambda IS - S^{-1}AS) = \det(S^{-1}(\lambda I - A)S) = \det(\lambda I - A) \text{ (from (3ii).}$$

Q.E.D.

Homework 11 due 29/1/07

(1) Find the eigenvalues and corresponding eigenvectors for $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

$(\lambda - 1)^2 - 9 = 0$: eigenvalues $4, -2$.

$$4 : \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \qquad -2 : \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix}$$

The cofactors of the top row are $[3\ 3]^T$ and $[-3\ 3]^T$ respectively; these are eigenvectors.

(2) Find the eigenvalues of

$$\begin{bmatrix} -10 & 21 & -6 \\ -4 & 9 & -2 \\ 10 & -18 & 7 \end{bmatrix}.$$

(Hint: they are integers, and the constant term in $\det(\lambda I - A)$ is -6).

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

(3) Find eigenvectors for the above matrix.

$$1 : \begin{bmatrix} 11 & -21 & 6 \\ 4 & -8 & 2 \\ -10 & 18 & -6 \end{bmatrix} \quad 2 : \begin{bmatrix} 12 & -21 & 6 \\ 4 & -7 & 2 \\ -10 & 18 & -5 \end{bmatrix} \quad 3 : \begin{bmatrix} 13 & -21 & 6 \\ 4 & -6 & 2 \\ -10 & 18 & -4 \end{bmatrix}$$

Taking cofactors of the first row, we get $[12\ 4\ -8]^T$, $[-1\ 0\ 2]^T$, $[-12\ -4\ 12]^T$.

(4) Find eigenvalues, **but not** eigenvectors, for the matrix

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

$$(\lambda-1)(\lambda-2)(\lambda-1)-9(\lambda-2)=(\lambda-2)((\lambda-1)^2-9)=(\lambda-2)(\lambda-4)(\lambda+2).$$

Eigenvalues $2, 4, -2$.

Homework 12 due 05/2/07

(1)(12 marks) The following matrix has eigenvalues $4, 2, -2$. Calculate corresponding eigenvectors.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 3 \end{bmatrix} : \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix} : \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 & 0 & -3 \\ 0 & -4 & 0 \\ -3 & 0 & -3 \end{bmatrix} : \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix}.$$

(2)(12 marks) The following matrix has repeated eigenvalues. Calculate its eigenvalues.

$$\begin{bmatrix} -9 & 20 & 6 \\ -5 & 11 & 3 \\ 1 & -2 & 0 \end{bmatrix} \qquad \begin{array}{l} \lambda(\lambda+9)(\lambda-11)-60-60-6(\lambda-11)+ \\ 6(\lambda+9)+100\lambda=\lambda(\lambda-1)^2. \end{array}$$

(3)(12 marks) Calculate a basis which brings the above matrix to upper triangular form.

For $\lambda=0$ and $\lambda=1$ we get $\lambda I - A =$

$$\begin{bmatrix} 9 & -20 & -6 \\ 5 & -11 & -3 \\ -1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 10 & -20 & -6 \\ 5 & -10 & -3 \\ -1 & 2 & 1 \end{bmatrix}$$

This gives two eigenvectors $v_1=[6\ 3\ -1]^T$ and $v_2=[-4\ -2\ 0]^T$. Extend this to a basis with the vector $v_3=[1\ 0\ 0]^T$ (say). ($[0\ 0\ 1]^T$ is not independent of the other two). Then

$$Av_1=0v_1, \quad Av_2=v_2, \quad \text{and} \quad Av_3=[-9\ -5\ 1]^T.$$

Checking the 'new' coordinates by solving

$$\begin{bmatrix} 6 & -4 & 1 \\ 3 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \\ 1 \end{bmatrix}$$

which is easily solved by substituting for a, then b then c, we get $a=-1, b=1, c=1$. In other words, $Av_3=-v_1+v_2+v_3$, and the coefficient of v_3 is the eigenvalue 1. (This calculation is unnecessary. It is only necessary that v_1, v_2, v_3 be linearly independent.)

(4)(14 marks) (i) What is the probability of a 'full house' (3 of one value and 2 of another) in poker?

$$(13)(12)\binom{4}{1}\binom{4}{2} \div \binom{52}{5} = 3744/2598960 = 0.0014405.$$

(ii) What is the probability that all six of your lotto numbers, from 1 to 42, are wrong?

$$\binom{36}{6} \div \binom{42}{6} = 0.3713059.$$

Homework 13 due 12/2/07

(1)(10 marks) Let A be the event 'Out of five coin-tosses, the number of heads is odd.' List the 16 points in A, in lexicographical order, beginning with HHHHH. (Thus if H=0 and T=1, and the numbers are interpreted in binary, we get ascending order.)

```
HHHHH HHHTT HHTHT HHTTH
HTHHT HTHTH HTTHH HTTTT
THHHT THHTH THTHH THTTT
TTHHH TTHTT TTTHT TTTTH
```

(2)(12 marks) Continuing question (1): for $0 \leq i \leq 5$, calculate the *conditional* probability (relative to A) that there are i heads, given A.

```
1: HTTTT THTTT TTHTT TTTHT TTTTH


3: HHHTT HHTHT HHTTH
HTHHT HTHTH HTTHH
THHHT THHTH THTHH
TTHHH


5: HHHHH
```

If i is even, the probability is zero. For 1 we get 5/16; for 3, 10/16; and for 5, 1/16.

(3)(12 marks) Cars arrive at a traffic light at a rate of 4 cars per length of red signal (cars arriving on green or orange go straight through). What is the probability that (i) exactly 5 (ii) at least 5 cars will be waiting at the lights when the lights change?

(i) $e^{-4}(4^5/120) = .1562934510$ (ii) $e^{-4}(1+4+4^2/2+4^3/6+4^4/24+4^5/120) = .6288369321$.

(4)(16 marks) A product is made on four assembly lines A,B,C,D, with relative rates of production $4:3:2:1$ and probabilities $5\%, 10\%, 10\%$, and 25% of being defective. Calculate the probabilities that a defective product came from (i) A, (ii) B, (iii) C, or (iv) D.

X	A	B	C	D
prob(X)	0.4	0.3	0.2	0.1
prob($D \mid X$)	0.05	0.1	0.1	0.25
prob(D and X)	0.02	0.03	0.02	0.025
prob($X \mid D$)	.2105	.3157	.2015	.2631

From the calculation, the probability of a defective is 0.95, so the answer is as shown.

Homework 14 due 19/2/07

(1)(12 marks). The following sample comes from identically distributed items. Estimate the mean and variance of the distribution.

```
0 7 6 2 0 7 0 9 8 2
```

Answer. Average 4.100000, estimates the mean. The estimated variance is 13.211111.

(2) (12 marks). Suppose you visit a city for the first time and someone tells you that the buses are numbered from 1 to N, but that person is unsure of what N is. The highest bus-number you see is 78. What is the maximum-likelihood estimator for N?

Answer. (We have to assume that all bus-numbers have the same probability of being seen.) The probability cannot be more than 1/78, so 78 is the maximum likelihood estimator.

(3) (12 marks). Suppose that the postal service were slightly unreliable, and letters had a certain probability p of not being delivered. To estimate p, suppose someone mailed 100 letters to himself during a year, and suppose that 94 were delivered. Find the maximum-likelihood estimate of p.

Answer. The distribution is binomial $B(100, p)$. From the notes, the maximum likelihood estimator of p is 6/100.

(4) (14 marks). There are r people in a room, none of them with birthdays on February 29th. What is the probability that all r people have different birthdays? (Assume their birthdays are independently distributed).

Answer. Arrange the birthdays $x_1, \ldots, x_r$ in any order. The probability that $x_1 \neq x_2$ is 364/365. Given $x_1 \neq x_2$, the probability that $x_2 \neq x_3$ is 363/365, and so on. The answer is

$$\frac{364}{365}\frac{363}{365}\cdots\frac{365-r+1}{365}$$

(or 1 if $r = 1$.)

Calculate the smallest r such that there is a better-than-evens chance that two or more people in the room have the same birthday.

Answer.

```
Prob. 19 birthdays differ = 0.620881
Prob. 20 birthdays differ = 0.588562
Prob. 21 birthdays differ = 0.556312
Prob. 22 birthdays differ = 0.524305
Prob. 23 birthdays differ = 0.492703
Prob. 24 birthdays differ = 0.461656
```

The answer is 23.

Homework 15 due 26/2/07

(1)(12 marks) Show by tables that (i) $X \implies Y$ and $Y \implies X$ are not equivalent, but (ii) $X \implies Y$ and $(\neg Y) \implies (\neg X)$ are equivalent.

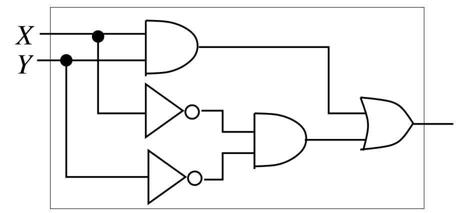
Answer. Comparing the columns in the truth-table below, $X \implies Y$ is equivalent to $(\neg Y) \implies \neg X$ but not to $Y \implies X$.

X	Y	$X \implies Y$	$Y \implies X$	$(\neg Y) \implies \neg X$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	0
1	1	1	1	1

(2)(8 marks) Complete the truth-table for circuit (i) below.

X	Y	$f(X,Y)$
0	0	1
0	1	0
1	0	0
1	1	1

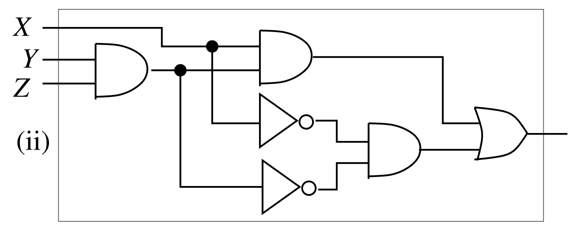
(i)



(3) (16 marks) and for circuit (ii):

X	Y	Z	$f(X,Y,Z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

(ii)



(4)(14 marks) Construct an expression, using only $\implies$ and $\neg$, which realises the function computed by circuit (ii) above. **Show the construction in steps.**

Answer: first, circuit (i) (not asked)

$$(X \wedge Y) \vee ((\neg X) \wedge \neg Y)$$
$$(\neg(X \implies \neg Y)) \vee \neg((\neg X) \implies Y)$$
$$(X \implies \neg Y) \implies \neg((\neg X) \implies Y).$$

Replace Y by $Y \wedge Z$.

$$(X \implies \neg(Y \wedge Z)) \implies \neg((\neg X) \implies (Y \wedge Z))$$
$$(X \implies \neg\neg(Y \implies \neg Z)) \implies \neg((\neg X) \implies \neg(Y \implies \neg Z)) : \quad \text{alternatively}$$
$$(X \implies (Y \implies \neg Z)) \implies \neg((\neg X) \implies \neg(Y \implies \neg Z)), \quad \text{also}$$
$$(X \implies (Y \implies \neg Z)) \implies \neg((Y \implies \neg Z) \implies X)$$

Homework 16 due 05/3/07

(1)(16 marks) Prove by resolution that the CNF

$$ABC, A\bar{B}C, A\bar{B}\bar{C}, B\bar{C}, \bar{A}BC, \bar{A}\bar{B}$$

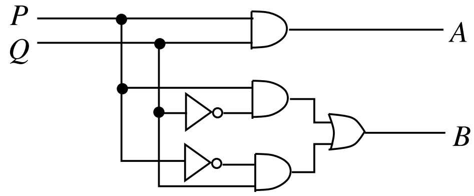
is inconsistent.

Answer. (There are many possible answers.)

1. $A\bar{B}\bar{C}, B\bar{C} \mapsto A\bar{C}$
2. $ABC, A\bar{C} \mapsto AB$
3. $A\bar{B}C, AB \mapsto AC$
4. $A\bar{B}\bar{C}, AC \mapsto A\bar{B}$
5. $AB, A\bar{B} \mapsto A$
6. $B\bar{C}, \bar{A}BC \mapsto \bar{A}B$
7. $\bar{A}\bar{B}, \bar{A}B \mapsto \bar{A}$
8. $A, \bar{A} \mapsto \Box$

(2) (16 marks) Construct a truth-table (with columns for inputs P, Q and outputs A, B). Hence say what it does. **Answer:** from the table below, it's a half-adder.

P	Q	A	B
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0



(3)(18 marks) Using the 3-bit addition circuit as a starting-point, design a circuit which converts input $P_2P_1P_0$ to its 2s complement, by negating all bits and adding 1. **Answer: see figure.**