Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science

School of Mathematics

JF Mathematics **Trinity Term 2018**
SF Theoretical Physics

Maths 1213 - Introduction to Group Theory

Wednesday, May 2 Sports Hall? 09:30 — 11:30

Prof. C. Ó Dúnlaing

Instructions to Candidates:

Attempt 3 questions.
Show all work.
Remember to fold down and glue the flap on every answer booklet.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let x, y be elements of a group G. Prove that $(xy)^{-1} = y^{-1}x^{-1}$.

   (b) Suppose that $x^2 = 1$ for every element x of a group G. Prove that G is abelian (commutative).

   (c) The symmetric difference operator Δ is defined as follows, on sets A, B:

$$A\Delta B = \{x : x \in A \cup B \text{ and } x \notin A \cap B\}.$$

   i. Prove that $x \in A\Delta(B\Delta C)$ if and only if x belongs *either* to exactly one of A, B, C, *or* to all three.

   ii. Prove that Δ is associative.

2. (a) Suppose that we are given a left action $\ell_g : x \mapsto gx$ of a finite group G on a finite set S. For each $x \in S$, O_x is the orbit of x and G_x is the subgroup fixing x. Prove

$$|O_x| = \frac{|G|}{|G_x|}.$$

   (b) The quaternion group Q_8 has elements $\pm 1, \pm i, \pm j, \pm k$. If we identify i with the unit vector $(1, 0, 0)$ in $\mathbb{R}^3$, similarly j, k, then Q_8 acts on $\mathbb{R}^3$ by conjugation, that is,

$$\ell_g : \quad xi + yj + zk \mapsto g(xi + yj + zk)g^{-1},$$

   is a left action on $\mathbb{R}^3$. (No proof required.)

   Calculate the orbit of $(1, 1, 2) = i + j + 2k$ under this action.

3. (a) Given $n \geq 2$, prove that the 2-cycles $\{(ij) : 1 \leq i < j \leq n\}$ generate the symmetric group S_n.

   (b) Prove that $(12), (1234)$ generate S_4.

   (c) It can be shown that $(123), (124), (134), (234)$ generate the alternating group A_4. Prove that $(123), (234)$ generates A_4.

4. (a) Define when a map $f : G \to H$ of groups is an *isomorphism*.

   (b) Prove that the composition of two isomorphisms $f : G \to H, g : H \to K$, is an isomorphism. (You may assume that the composition and inverse of bijective maps are bijective.)

   (c) Prove that the inverse of an isomorphism is an isomorpism.

   (d) Divide the following four groups into classes, so that groups in the same class are isomorphic and groups in different classes are not. Give reasons for your answer. (i) $\mathbb{Z}_6$ (ii) $\mathbb{Z}_9^*$ (iii) S_3 (iv) $\mathbb{Z}_7^*$.