Deriving the Electromagnetic Field from the Liénard–Wiechert potential

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Given the path of a particle of charge q located at space-time coordinate $x^\mu_{\mathrm{q}}(\tau)$ for invariant time τ, the Liénard–Wiechert four potential A^ν at coordinate x^μ, $\mu \in \{0,\, 1,\, 2,\, 3\}$, induced by the motion of the charge can be shown to be

$$\frac{4\pi}{q}\, A^\nu = \frac{V^\nu}{R^\sigma V_\sigma} \tag{1}$$

where the expression on the right is evaluated at the retarded time τ_0 defined by

$$R^\sigma R_\sigma = 0 \tag{2}$$

and where the argument of the potential relative to the location of the charge is

$$R^\sigma \;=\; x^\sigma - x^\sigma_q(\tau)\,. \tag{3}$$

To find the electromagnetic field $F^{\mu\nu}$ at the coordinate x^ρ resulting from the expression for the four potential exhibited in equation (1) consider the relative position four vector $R^\sigma = R^\sigma(x^\rho, \tau)$ as a function of x^ρ and τ. Since the path of the particle, $x^\mu_{\mathrm{q}}(\tau)$, intersects the past lightcone of the event x^ρ once we can regard the invariant time, $\tau = \tau(x^\rho)$, as a function of x^ρ. Equation (3) provides

$$\partial_\mu R^\sigma \;=\frac{\partial R^\sigma}{\partial x^\mu} \;=\; \frac{\partial x^\sigma}{\partial x^\mu} \;-\; \frac{\mathrm{d}x^\sigma_q}{\mathrm{d}\tau}\,\frac{\partial\,\tau}{\partial x^\mu} \;=\; \delta^\sigma_\mu \;-\; V^\sigma\,\frac{\partial\,\tau}{\partial x^\mu} \tag{4}$$

involving a Kronecker delta and a four velocity of the charged particle denoted by

$$V^\sigma \;=\; \frac{\mathrm{d}\,x^\sigma_q}{\mathrm{d}\,\tau}\,. \tag{5}$$

Again, taking the partial derivative of equation (2) with respect to x^μ yields

$$(\partial_\mu R^\sigma)\,R_\sigma \;+\; R^\sigma(\partial_\mu R_\sigma) \;=\; 2\,R_\sigma(\partial_\mu R^\sigma) \;=\; 0 \tag{6}$$

from which, with an abbreviated from of equation (4), one may conclude that

$$R_\sigma\left(\delta^\sigma_\mu - V^\sigma\partial_\mu\tau\right) \;= R_\mu - R_\sigma V^\sigma \partial_\mu\tau \;=\; 0 \tag{7}$$

resulting in equivalent expressions for the four gradient of the invariant time

$$\partial_\mu\tau \;=\; \frac{R_\mu}{R_\sigma V^\sigma}\,, \quad \text{that is,} \quad \frac{\partial\,\tau}{\partial x_\mu} \;=\; \frac{R^\mu}{R^\sigma V_\sigma}\,. \tag{8}$$

Taking the partial derivative of the four potential below will require the derivative

$$\partial^\mu R_\rho \;=\; \delta^\mu_\rho \;-\; V_\rho \,\frac{\partial\,\tau}{\partial x_\mu} \;=\; \delta^\mu_\rho \;-\; V_\rho \,\frac{R^\mu}{R^\sigma V_\sigma} \qquad (9)$$

that follows from the use of equations (4) and (8), written with altered indices. Also required will be a derivative of the four velocity $V^\nu = V^\nu(\tau)$ in equation (5), treated as a function of $\tau(x^\rho)$, and again using equation (8)

$$\partial^\mu V^\nu \;=\; \frac{\partial V^\nu}{\partial x_\mu} \;=\; \frac{\mathrm{d}V^\nu}{\mathrm{d}\tau}\,\frac{\partial \tau}{\partial x_\mu} \;=\; \dot V^\nu \,\frac{R^\mu}{R^\sigma V_\sigma} \;=\; \frac{R^\mu\,\dot V^\nu}{R^\sigma V_\sigma} \qquad (10)$$

where $\dot V^\mu$ refers to $\mathrm{d}V^\mu/\mathrm{d}\tau$, the derivative with respect to the invariant time. Evaluating the partial derivative of the four potential A^ν in equation (1) reveals

$$\begin{aligned}
\frac{4\pi}{q}\,\partial^\mu A^\nu \;&=\; \frac{\partial^\mu V^\nu}{R^\sigma V_\sigma} - \frac{V^\nu}{(R^\sigma V_\sigma)^2}\left[R_\rho\,(\partial^\mu V^\rho) \,+\, (\partial^\mu R_\rho)\,V^\rho\,\right] \qquad (11)\\
&=\; \frac{R^\mu\,\dot V^\nu}{(R^\sigma V_\sigma)^2} - \frac{V^\nu}{(R^\sigma V_\sigma)^2}\left[R_\rho\,\frac{R^\mu\,\dot V^\rho}{R^\lambda V_\lambda} \,+\, \left(\delta^\mu_\rho \,-\, V_\rho\,\frac{R^\mu}{R^\lambda V_\lambda}\right)V^\rho\right]\\
&=\; \frac{R^\mu \dot V^\nu}{(R^\sigma V_\sigma)^2} - \frac{R_\rho \dot V^\rho}{(R^\sigma V_\sigma)^3}\,R^\mu V^\nu \,-\, \frac{V^\mu V^\nu}{(R^\sigma V_\sigma)^2} \,+\, \frac{V_\rho V^\rho}{(R^\sigma V_\sigma)^3}\,R^\mu V^\nu\,.
\end{aligned}$$

The third term will not contribute to $F^{\mu\nu}$, being symmetric in indices μ and ν. Noting that the four velocity of the particle $V^\sigma = \gamma(c,\,\vec v)$ obeys

$$V_\rho V^\rho \;=\; \gamma^2(c^2 - \vec v^{\,2}) \;=\; \gamma^2 c^2 \gamma^{-2} \;=\; c^2 \qquad (12)$$

equations (11) and (12) indicate that the electromagnetic field may be written

$$F^{\mu\nu} \;=\; \partial^\mu A^\nu - \partial^\nu A^\mu \;=\; F^{\mu\nu}_{\mathrm{acc}} + F^{\mu\nu}_{\mathrm{vel}} \qquad (13)$$

where the acceleration and velocity elements of the electromagnetic tensor are

$$4\pi\,F^{\mu\nu}_{\mathrm{acc}} \;=\; \frac{q}{(R^\sigma V_\sigma)^2}(R^\mu \dot V^\nu \,-\, R^\nu \dot V^\mu) - \frac{q\,R^\rho \dot V_\rho}{(R^\sigma V_\sigma)^3}\,(R^\mu V^\nu - R^\nu V^\mu) \qquad (14)$$

$$4\pi\,F^{\mu\nu}_{\mathrm{vel}} \;=\; \frac{q\,c^2}{(R^\sigma V_\sigma)^3}\,(R^\mu V^\nu - R^\nu V^\mu) \qquad (15)$$

with all quantities here being evaluated at the retarded time given by equation (2). In the case where the particle is at rest with $V^\sigma = (c, \vec 0)$ the radiative part of the field is zero $F^{\mu\nu}_{\mathrm{acc}} = 0$; the non-radiative part indicates here that the electric field $F^{i\,0}_{\mathrm{vel}}$ gives the expected inverse square law and that the magnetic field $F^{i\,j}_{\mathrm{vel}}$ is zero.

References

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