STUDY WEEK EXERCISES FOR COURSE MA3431

Answers are due at the first MA3431 lecture after Study Week

Recommended textbooks for the Course MA3431 in Classical Field Theory

- Classical Electrodynamics, J. D. Jackson, John Wiley, 1998 (3rd ed) [537.12 K23]
- The Classical Theory of Fields, E. M. Lifshitz and L. D. Landau [530.14 L52]
- Classical Field Theory, Francis E. Low, John Wiley & Sons, 1997 [530.14 N71]
- Classical Mechanics, Herbert Goldstein, Addison-Wesley, 1980 [531 J0*1]

1. A Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ depends upon a scalar field $\phi(x)$ and its partial derivative $\partial_\mu \phi = \partial \phi / \partial x^\mu$. The space-time point x refers to x^μ with components (x^0, x^1, x^2, x^3). Consider the following Lagrangian density where σ is a constant

$$\mathcal{L} \;=\; \tfrac{1}{2}\, \partial_\lambda \phi \, \partial^\lambda \phi \;+\; \tfrac{1}{5}\, \sigma \, \phi^5 \, .$$

   (a) Explain Einstein summation and exhibit $\partial_\lambda \phi \, \partial^\lambda \phi$ in terms of its components.

   (b) Write down the Euler-Lagrange equation of motion of the scalar field $\phi(x)$.

   (c) Evaluate the equation of motion for the particular Lagrangian density $\mathcal{L}$ provided above.

   (d) Calculate from the above Lagrangian density $\mathcal{L}$ the following stress tensor

$$T^{\mu\nu} \;=\; \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \, \partial^\nu \phi \;-\; g^{\mu\nu} \mathcal{L}$$

   where the metric tensor $g^{\mu\nu}$ is diagonal with matrix entries $(+1, -1, -1, -1)$.

   (e) Find the 4-divergence of the stress tensor $T^{\mu\nu}$ you obtained, namely, determine

$$\partial_\mu T^{\mu\nu} \, .$$

   (f) Show that the stress tensor $T^{\mu\nu}$ is conserved by demonstrating that its 4-divergence is zero when the scalar field $\phi(x)$ obeys its equation of motion

$$\partial_\mu T^{\mu\nu} = 0 \, .$$

2. (a) Show that the electric and magnetic induction fields transform according to

$$\begin{aligned} E_1' &= E_1 & B_1' &= B_1 \\ E_2' &= \gamma\, E_2 - \gamma\beta\, B_3 & B_2' &= \gamma\, B_2 + \gamma\beta\, E_3 \\ E_3' &= \gamma\, E_3 + \gamma\beta\, B_2 & B_3' &= \gamma\, B_3 - \gamma\beta\, E_2 \end{aligned}$$

namely, Eq. (11.148) of J. D. Jackson's *Classical Electrodynamics*, using

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho\, \Lambda^\nu{}_\sigma\, F^{\rho\sigma}$$

when the velocity βc of frame K' is directed along the x^1 axis of frame K with

$$\Lambda^\mu{}_\rho = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Assume that the elements of the matrix $F_{\mu\nu}$ may be written in terms of the fields $\vec{E}$ and $\vec{B}$ in a form given by Eq. (11.137) of J. D. Jackson's *CED*, viz.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

(b) Show that the electric and magnetic fields in frame K of the electrostatic field of a stationary charge q in a frame K' moving with velocity $v = \beta c$ along the x^1 axis of frame K are given by Eq. (11.152) of J. D. Jackson's *CED, 3e*

$$\begin{aligned} E^1 &= -\frac{\gamma q v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} & B^1 &= 0 \\ E^2 &= \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} & B^2 &= 0 \\ E^3 &= 0 & B^3 &= \frac{\beta\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \end{aligned}$$

t referring to the time since the origins of the frames K and K' overlapped and b being the closest distance of approach of the charge, assumed fixed on the x'^2 axis. J. D. Jackson uses ordinary Cartesian spatial components in Eq. (11.148) and in Eq. (11.152). Contravariant indices are used here.