CLASSICAL FIELD THEORY AND ELECTRODYNAMICS

Exercises and Recommended Texts for Course 432

1. (a) Show that the electric and magnetic induction fields transform according to Eq. (11.148) of J. D. Jackson's *Classical Electrodynamics, 3e*, using

$$F'^{\mu\nu} \;=\; \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\,F^{\rho\sigma}$$

when the velocity βc of the frame K' is directed along the x^1 axis of frame K with

$$\Lambda^{\mu}{}_{\rho} \;=\; \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   (b) Show that the electric and magnetic fields in frame K of the electrostatic field of a stationary charge q in a frame K' moving with velocity $v = \beta c$ along the x_1 axis of frame K are given by Eq. (11.152) of J. D. Jackson's *CED, 3e*.

$$E_1 \;=\; -\frac{\gamma q v t}{(b^2 + \gamma^2 v^2 t^2)^{\frac{3}{2}}}$$

$$E_2 \;=\; \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{\frac{3}{2}}}$$

$$B_3 \;=\; \frac{\beta\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{\frac{3}{2}}}$$

with the other components vanishing, t being the time since the origins of the frames K and K' overlapped and b referring to the closest distance of approach of the charge, assumed fixed on the x_2' axis.

2. An alternative Lagrangian density for the electromagnetic field due to Enrico Fermi is

$$\mathcal{L} \;=\; -\frac{1}{8\pi}\,\partial_\mu A_\nu\,\partial^\mu A^\nu - \frac{1}{c}\,J_\mu\,A^\mu\,.$$

   (a) Derive the Euler-Lagrange equations of motion. Under what assumptions are they the Maxwell equations of electrodynamics?

   (b) Show explicitly, and with what assumptions, that Fermi's Lagrangian density differs from

$$\mathcal{L} \;=\; -\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} \;-\; \frac{1}{c}\,J_\mu\,A^\mu$$

by a 4-divergence. Does the added 4-divergence affect the action? Does it affect the equations of motion? [J. D. Jackson, Problem 12.13 (2e), 12.14 (3e)]

3. In 1930 Proca introduced a Lagrangian density for a massive vector field in interaction with an external source J^ν

$$\mathcal{L}_{\text{Proca}} \;=\; -\frac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} + \frac{m^2}{8\pi}\, A_\mu\, A^\mu - \frac{1}{c}\, J_\mu\, A^\mu$$

where the Compton wave number $m = m_\gamma c/\hbar$ relates to the mass m_γ of the field.

   (a) Show that the symmetric stress energy momentum tensor for the Proca fields is

$$4\pi\Theta^{\mu\nu} = g^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + m^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A_\rho A^\rho\right) .$$

   (b) Demonstrate that the Proca equations of motion for the massive field are

$$\partial_\mu F^{\mu\nu} + m^2\, A^\nu = \frac{4\pi}{c}\, J^\nu \, .$$

   (c) Verify that the differential conservation laws take the same form as for massless electromagnetic fields, namely,

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c} J_\rho F^{\rho\nu} \, .$$

   (d) Show explicitly that the time-time and space-time components of the Proca symmetric stress energy momentum tensor are [Jackson, Problem 12.16 (3e)]

$$\begin{aligned} 8\pi\Theta^{00} &= \vec{E}^2 + \vec{B}^2 + m^2\left(A^0A^0 + \vec{A}\cdot\vec{A}\right) \\ 4\pi\Theta^{i\,0} &= \left(\vec{E}\times\vec{B}\right)^i + m^2\, A^i\, A^0 \, . \end{aligned}$$

4. Other problems of J. D. Jackson from 12.12 (2e) onwards.

Recommended texts for Classical Field Theory and Electrodynamics.

- Classical Electrodynamics, J. D. Jackson, John Wiley, 1998 (3rd ed) [537.12 K23]
- The Classical Theory of Fields, E. M. Lifshitz and L. D. Landau [530.14 L52]
- Classical Field Theory, Francis E. Low, John Wiley & Sons, 1997 [530.14 N71]
- Classical Mechanics, Herbert Goldstein, Addison-Wesley, 1980 [531 J0*1]