STUDY WEEK EXERCISES FOR COURSE MA3432

Recommended Textbooks for Course MA3432 in Classical Electrodynamics

- Classical Electrodynamics, J. D. Jackson, John Wiley, 1998 (3rd ed) [537.12 K23]
- The Classical Theory of Fields, E. M. Lifshitz and L. D. Landau [530.14 L52]
- Classical Field Theory, Francis E. Low, John Wiley & Sons, 1997 [530.14 N71]
- Classical Mechanics, Herbert Goldstein, Addison-Wesley, 1980 [531 J0*1]

1. An alternative Lagrangian density for the electromagnetic field due to Enrico Fermi is
$$\mathcal{L} \,=\, -\frac{1}{2}\,\partial_\mu A_\nu\,\partial^\mu A^\nu - \frac{1}{c}\,J_\mu\,A^\mu\,.$$
   (a) Derive the Euler-Lagrange equations of motion. Under what assumptions are they the Maxwell equations of electrodynamics?

   (b) Show explicitly, and with what assumptions, that the Lagrangian density of Fermi differs from
$$\mathcal{L} \,=\, -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} \,-\, \frac{1}{c}\,J_\mu\,A^\mu$$
   by a 4-divergence. Does the added 4-divergence affect the action? Does it affect the equations of motion? [J D Jackson, Problem 12.13 (2e), 12.14 (3e)].

2. Consider a charge q whose velocity $c\vec{\beta}$ is parallel to its acceleration $c\vec{\alpha}$. If θ is the angle the emitted radiation makes with the common direction of the velocity and acceleration, the power emitted by a charge q at very high energy with large γ factor may be written in the approximate form
$$\frac{dP'}{d\,\Omega} \,=\, \frac{q^2\,\alpha^2\,\gamma^8}{\pi^2\,c}\,\frac{2\,\gamma^2\,\theta^2}{(\gamma^2\,\theta^2\,+\,1\,)^5}\,.$$
   (a) Evaluate the total power radiated to all angles by an ultra-relativistic particle of charge q, keeping the leading power of γ only.

   (b) Compare this total power generated by a high energy, linearly accelerating charge q, with that expected from Liénard's expression for the power radiated
$$P \,=\, \frac{q^2\,\gamma^6}{6\,\pi\,c}\,\left[\,\vec{\alpha}^{\,2} \,-\, (\vec{\alpha}\times\vec{\beta}\,)^2\right].$$

3. In 1930 Proca introduced a Lagrangian density for a massive vector field in interaction with an external source J^ν

$$\mathcal{L}_{\text{Proca}} \;=\; -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} + \frac{m^2}{2}\,A_\mu\,A^\mu - \frac{1}{c}\,J_\mu\,A^\mu$$

where the Compton wave number $m = m_\gamma c/\hbar$ relates to the mass m_γ of the field.

(a) Show that the symmetric stress energy momentum tensor for Proca fields is

$$\Theta^{\mu\nu} = g^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + m^2\left(A^\mu A^\nu - \frac{1}{2}g^{\mu\nu}A_\rho A^\rho\right)\,.$$

(b) Demonstrate that the Proca equations of motion for the massive field are

$$\partial_\mu F^{\mu\nu} + m^2\,A^\nu = \frac{1}{c}\,J^\nu\,.$$

(c) Verify that the differential conservation laws take the same form as for massless electromagnetic fields, namely,

$$\partial_\mu \Theta^{\mu\nu} = \frac{1}{c}J_\rho F^{\rho\nu}\,.$$

(d) Show explicitly that the time-time and space-time components of the Proca symmetric stress energy momentum tensor are [Jackson, Problem 12.16 (3e)]

$$\begin{aligned} 2\,\Theta^{00} &= \vec{E}^2 + \vec{B}^2 + m^2\left(A^0A^0 + \vec{A}\cdot\vec{A}\right) \\ \Theta^{\,i\,0} &= \left(\vec{E}\times\vec{B}\right)^i + m^2\,A^i\,A^0\,. \end{aligned}$$

Recommended Textbooks for Course MA3432 in Classical Electrodynamics

- Classical Electrodynamics, J. D. Jackson, John Wiley, 1998 (3rd ed) [537.12 K23]
- The Classical Theory of Fields, E. M. Lifshitz and L. D. Landau [530.14 L52]
- Classical Field Theory, Francis E. Low, John Wiley & Sons, 1997 [530.14 N71]
- Classical Mechanics, Herbert Goldstein, Addison-Wesley, 1980 [531 J0*1]

Classical Electrodynamics <nhb@maths.tcd.ie> Dr. Buttimore