The Life and Work of A. A. Markov

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KEY WORDS: Andrei A. Markov biography, Markov chain, weak law of large numbers

ABSTRACT

The Russian mathematician A. A. Markov (1856–1922) is known for his work in number theory, analysis, and probability theory. He extended the weak law of large numbers and the central limit theorem to certain sequences of dependent random variables forming special classes of what are now known as Markov chains. For illustrative purposes Markov applied his chains to the distribution of vowels and consonants in A. S. Pushkin's poem "Eugeny Onegin". At present, much more important applications of Markov chains have been discovered. Here we present an overview of Markov's life and his work on the chains.

1. Introduction

Andrei Andreevich Markov was a gifted Russian mathematician, a disciple of the renowned Pafnuty Lvovich Chebyshev. At the age of 30 Markov became a professor at St. Petersburg University and a member of St. Petersburg Academy of Sciences. He published more than 120 scientific papers on number theory, continuous fraction theory, differential equations, probability theory, and statistics. His classical textbook, "Calculus of Probabilities," was published four times in Russian and was translated into German.

Many of his papers were devoted to creating a new field of research, Markov chains. The solution of many fundamental problems of modern science and technology would not be possible without his contributions. It is fitting that these chains bear his name, acknowledging his trailblazing role in the development of Markov chains. In fact, as early as 1926, just

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twenty years after his initial discoveries, a paper by Russian mathematician S. N. Bernstein used the phrase "Markov chain" [3].

The present paper is divided into three main sections. In the first section, we set the background for Markov's coming by describing the events prior to his life that had a great influence on his later work. In the middle section, we describe Markov's personal life, his childhood, his family, his wife, and his colleagues. The final section is devoted to Markov's academic work, with special emphasis on his chains. In addition to the references, we supply a list of the publications reviewing Markov's life and work.

2. Events in Russia Prior to A. A. Markov

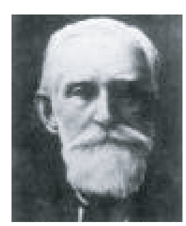
To provide the historical context of the life and work of Markov, we go back two centuries before his birth, to the time when Russia was ruled by the Czar and Emperor Peter the First, who has been called the Great Reformer of Russia.

This absolute monarch held power from 1694 until his death in 1725. There are many legendary stories about the terror Peter instilled with his horrific methods of torture and brutality towards nonconforming subjects. Paradoxically though, in many other ways Peter was very forward-thinking for his day. Peter travelled to the West often, visiting for extended periods with rulers from other "more advanced" societies. As a result of his travels, Peter initiated the great reform of Russia, patterned after the best of the West. His goal was to bring the slow, late-blooming Russia from the Middle Ages into the Enlightenment. In order to facilitate this westernized reform, Peter moved the capital from Moscow to St. Petersburg, the latter being much more accessible from the West. There he also established an academy of science. Having been a member of the Parisian Academy of Sciences, Peter saw the potential impact, direct and indirect, that science could have in the development of his country. On January 24, 1724 Peter issued an edict declaring the establishment of the St. Petersburg Academy of Sciences, which was to include both an academy and a gymnasium. Peter's Academy did not merely mimic the western academies he had studied, his academy was to be different in mission. It was not just a research body like many other academies, it was also an educational institution, hence the attached schools.

Among the early Academy members were Leonard Euler, Jakob Hermann, Nicholas and Daniel Bernoulli, and Christian Goldbach. There were three types of positions for acting members of the Academy working in St. Petersburg: adjunct, extraordinary academician, and ordinary academician, each garnering a corresponding salary. There were also three other types of positions. For Russian scientists living outside St. Petersburg, a corresponding member position was established. Foreign scientists could become foreign members, while members of the Royal Family and distinguished members of society were eligible for honorary membership. St. Petersburg Academy of Sciences has undergone many name changes since its establishment in 1724. Today it continues under the title of the Russian Academy of Sciences.

St. Petersburg Academy quickly became the hub for scientific advances in many fields. In fact, we can trace the roots of Markov's field of probability theory through several Academy members and their works. The Academy was the place of many Russian firsts in probability theory. In 1738, Daniel Bernoulli wrote the first Russian paper on probability theory, "Exposition of a new theory on the measurements of risk" [1]. Over a century later in 1846, V. Y. Bunyakovsky wrote the first Russian textbook on probability theory, "Fundamentals of the Mathematical Theory of Probability" [6]. That same year Pafnuty L. Chebyshev completed the first Russian dissertation on probability theory with his work at Moscow University, entitled "An Experience in an Elementary Analysis of the Probability Theory" [7]. P. L. Chebyshev started teaching at St. Petersburg University in 1847, and in 1860, he succeeded in V. Y. Bunyakovsky's footsteps, teaching a course on probability theory. It was at this time that the famous St. Petersburg Mathematical School originated, which later received renown from the works of P. L. Chebyshev and his students Y.-K. A. Sokhotsky, A. A. Markov, A. M. Lyapunov, G. F. Voronoi and others—an impressive list of students for any educator.

Markov, in studying under Chebyshev, was clearly influenced by his mentor both as a researcher and as a teacher. As a researcher, Chebyshev's influence on Markov can easily be traced. Chebyshev's influence began already before their first face-to-face encounter in 1874 when the young Markov began his studies at St. Petersburg University. We can trace Chebyshev's influence back to 1867, when Chebyshev's paper "On Mean Values" was published [8], which generalized Poisson's theorem on the weak law of large numbers. Chebyshev had initiated his systematic study of sequences of independent random variables, a study that his student Markov later broadened to include certain types of dependent random variables. Twenty years later in 1887, Chebyshev published another seminal paper, "On Two Theorems concerning Probability," which generalized the central limit theorem and presented the method of moments [9]. The proof of the central limit theorem by the method of moments was later completed by Markov. The famous probabilist Kolmogorov, who followed in the footsteps of Chebyshev and Markov, remarked that "Chebyshev was the first to estimate clearly and make use of such notions as random quantity and its expectation value" [24].



In addition to his research achievements, Chebyshev was also a remarkable teacher. Later we will see traces of Chebyshev's teaching style in Markov's style. Another of Chebyshev's students, A. M. Lyapunov, said of his teacher [24],

> His courses were not voluminous, and he did not consider the quantity of knowledge delivered; rather, he aspired to elucidate some of the most important aspects of the problems he spoke on. These were lively, absorbing lectures; curious remarks on the significance and importance of certain problems and scientific methods were always abundant. Sometimes he made a remark in passing, in connection with some concrete case they had considered, but those who attended always kept it in mind. Consequently, his lectures were highly stimulating; students received something new and essential at each lecture; he taught broader views and unusual standpoints.

3. Markov's Personal Life

Andrei Andreevich Markov was born on June 14, 1856 in the town of Ryazan, where his father, Andrei Grigorievich Markov, worked as a public officer at the Forestry Department. In the early 1860s, the Markovs moved to St. Petersburg, where Andrei entered a classical gymnasium in 1866. He was rather poor in many subjects, but not in mathematics, toward which he showed enthusiasm and which he even studied on his own. For some time he believed he had invented a new method for the solution of linear differential equations. He informed A. N. Korkin and E. I. Zolotarev, prominent Russian mathematicians of the time, about his "discovery", but they explained to the young man that the method, was in fact not new. Nevertheless, his discovery was important as it initiated his lasting relationship with Korkin and Zolotarev, two professors at the St. Petersburg University.

Markov graduated from the gymnasium in 1874 and entered the Faculty of Mechanics and Mathematics of St. Petersburg University, where Chebyshev, Korkin and Zolotarev lectured. In addition to lecturing at the university, Korkin and Zolotarev conducted special courses for the best students, which were held chiefly at their own homes. Markov was the most active attendee of these courses. In 1877, he was awarded a gold medal for his research on the topic, proposed by the faculty, "On Solution of Differential Equations With the Help of Continued Fractions".

In 1880, Markov defended his master's thesis, "On the Binary Square Forms with Positive Determinant", and, in 1884, his Doctorate, "On Certain Applications of the Algebraic Continuous Fractions". In 1886, Markov, by Chebyshev's proposal, was elected for his significant services to science as an adjunct of St. Petersburg Academy of Sciences. In 1890, he was promoted to extraordinary academician, and in 1896, ordinary academician.

In 1880, as an associate professor at the university, Markov lectured on "Introduction to Analysis" and "Differential and Integral Calculus". After Chebyshev resigned from the university in 1883, Markov started teaching the probability theory course. In 1886, he was elected as an extraordinary professor, and in 1893, as an ordinary professor. In 1905, celebrating the 25th anniversary of his work at St. Petersburg University, Markov was awarded an honorary professorship. Shortly after that, he retired from the university but continued to lecture on probability theory and the theory of continuous fractions.

At an Academy session in 1889, Markov read his paper, "On a question by D. I. Mendeleev" [12]. This paper appeared the following year. It contained the proof of the now famous Markov inequality for algebraic polynomials. In 1900, the first edition of his textbook, "Calculus of Probabilities" was published [14]. This successful text was subsequently published in three other editions in 1908, 1913, and 1924. It was also translated into German and published in 1912. Incidentally, it is possible to buy a copy of this German historical text from an online purveyor of historical texts. The going rate in 2003 was 450 Euros.

Much can also be said about Markov the teacher. He paid great attention to the way mathematics was taught at schools and vigorously protested against what he called harmful experiments carried out in that field. When the mathematics teacher who taught at his son's college suddenly retired, Markov volunteered to lecture on mathematics. He emphasized problem-solving, and for those who were eager to improve their knowledge, he granted additional classes on Sundays and during the vacation time. Perhaps Markov was influenced by the tradition of his teachers, Korkin and Zolotarev. This tradition of extracurricular studies was certainly one he thoroughly enjoyed and benefited from. Markov's teaching was so acclaimed that fellow mathematician Gunther proclaimed, "I know of instances where senior students ... would attend [his lectures] for the second time ... [even] after they had successfully passed Markov's course" [30]. Finally, the following quote from Markov himself gives not only his opinion on teaching, but also shows the influence of his mentor Chebyshev [30]:

> The alleged opinion that studies in seminars [in classes] are of the highest scientific nature, while exercises in solving problems are of the lowest [rank], is unfair. Mathematics to a considerable extent consists in solving problems, [and] together with proper discussion, [this] can be of the highest scientific nature while studies in ... seminars might be of the lowest [rank].

Aside from his academic career, Markov led a very active life. In fact, he was a political activist. Perhaps A. A.'s tendencies toward activism were innate, as his rebel tendencies apparently appeared in his school days. By adulthood, Markov was actively involved in many political and social issues. As mentioned earlier, honorary membership to St. Petersburg Academy of Sciences was often bestowed on members of the Royal Family and distinguished members of the society. Markov opposed such honorary membership for royals, whom he considered to have not earned such an honor. In fact, Markov refused to accept tsarist awards in protest against the exclusion of the esteemed writer A. M. Gorky from the Academy. Markov's protest grew to outrage when the nobleman Duke Dundook was unjustifiably, in Markov's opinion, accepted into the Academy. Markov wrote a distasteful limerick about the situation, which was dubbed unfit for a lady's ears. During the period from 1904-1915, Markov continued his written protestations. He wrote over 20 letters to newspapers about burning social and educational issues. As a result of these letters, the press referred to him by such colorful nicknames as "Andrew the Furious" and "the militant academician" [30]. In 1907, Markov publicly renounced his membership in the electorate when the government dissolved the new Parliament. In a rather humorous letter dated 1912, Markov in utmost gravity and sincerity wrote to the Most Holy Synod asking to be excommunicated from the Russian Orthodox Church. This was in response to the fact that earlier that year the Holy Synod had excommunicated Leo Tolstoy. Markov apparently wanted like treatment. Perhaps somewhat more humorous, the Synod replied in equal formality, "[Markov] has seceded from God's Church and [we] expunged him from the lists of Orthodox believers" [30].

Markov's protests were successful at times. For example, A. M. Gorky was eventually admitted into the Academy. Gorky and Markov would attend many Academy sessions together, but one rather bizarre incident stands out. Grodensky, biographer of Markov, recounts the following story from the last year of Markov's life, some time after the revolution of 1917 [28].

> On the 5th of March 1921, A. A. Markov communicated that on account of the absence of footwear he is not able to attend meetings of the Academy. A few weeks later the KUBU (Committee for Improvement of the Existence of Scientists), meeting under the chairmanship of A. M. Gorky, fulfilled the prosaic request of the famous mathematician. Time, however, provided a colorful sequel, of sorts, to this. At the meeting of the physico-mathematical section of the Academy of Science on the 25th May, Andrei Andreevich announced: "Finally, I received footwear; not only, however, is it stupidly stitched together, it does not in essence accord with my measurements. Thus, as before, I cannot attend meetings of the

> Academy. I propose placing the footwear received by me in the Ethnographic Museum as an example of the material culture of the current time, to which end I am ready to sacrifice it."

One final remark in this section on Markov's militance is noteworthy. Throughout his professional career, Markov maintained a very public animosity toward his colleague Pavel Alekseevich Nekrasov, whose work he considered an "abuse of mathematics." Nekrasov was originally a theologian by training, and later took up mathematics, eventually obtaining a professorship at Moscow University. Nekrasov was also an active member of the Moscow Mathematical Society, another prestigious mathematical school in Russia. The Moscow school often times maintained a clear tension with its sister school in St. Petersburg. Most mathematicians of the Moscow School were stout members of the Russian Orthodox Church and strong proponents of the religious doctrine of free will. Moscow members like Nekrasov tried to enlist statistics and probability to provide a foundation for their doctrine of free will. Of course, Markov, an atheist and eventual excommunicate of the Church quarreled endlessly with his equally outspoken counterpart Nekrasov. The disputes between Markov and Nekrasov were not limited to mathematics and religion, they quarreled over political and philosophical issues as well. However, their mathematical disagreements are most pertinent to this paper. Later in this paper, we reveal how Markov's animosity toward Nekrasov sparked his research on chains.

During his day, Markov was considered to be one of the best chess players in St. Petersburg, and often competed by correspondence. However, his most enduring match was his marriage to his wife Maria. During his father's placement as a manager in the estate of E. A. Valvatiev, Markov became acquainted with the Valvatiev's daughter, Maria. Later on, while still a student, Markov was invited to teach Maria mathematics. Soon after, he asked Valvatiev for permission to marry his daughter, but did not receive permission until 1883, and the marriage took place in the same year. Their son, Andrei Andreevich Markov, Junior (1903-1979), became a prominent Russian mathematician and worked in the fields of algebra, topology, mechanics, and mathematical logic. From 1959 to 1979, the junior Markov chaired the Department of Mathematical Logic at Moscow State University. After Markov Junior's death the Department was headed by A. N. Kolmogorov, who many consider the founder of the general theory of Markov processes, and who later chaired the Department of Probability Theory at Moscow State University from 1935 to 1966.

Markov grew up in a large family. In fact, he appears to have had much in common with one brother, Vladimir. A. A. Markov's father married twice. With his first wife, Nadezhda Petrovna, he had 4 sons and two daughters: Peter, Paul (died as a child), Maria, Eugenia, Andrei and Mikhail. With his second wife, Anna Josephovna, he had a son, Vladimir, and two daughters, Lydia and Catherine. A. A. Markov's brother, Vladimir Andreevich Markov, born in 1871, also studied in the Department of Mechanics and Mathematics of St. Petersburg University. In 1892, Vladimir extended his brother's 1889 inequality for algebraic polynomials to consider all derivatives. Unfortunately, the Markov brothers' mathematical collaboration was cut short; Vladimir died of tuberculosis at the age of 26. After Vladimir's death in 1897, A. A. Markov completed and published Vladimir's unfinished master's thesis.

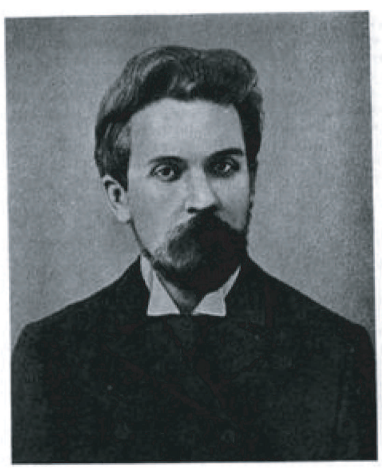
For most of his childhood Markov used crutches due to a severe inborn deformity of his knee. At the age of ten, he had an operation on his knee and afterwards he was able to walk only with a slight limp. Later in life he developed an aneurysm in his leg, which led to multiple surgeries throughout the years. One of the operations on his leg was fatal; on July 20, 1922, A. A. Markov died of sepsis. He was buried in the Mytrophany Cemetery in St. Petersburg.

In the next few pages we include a brief photo gallery of Markov and his family [32]. Below there are four photos. The upper right shows a young Markov, most likely in his twenties. The upper right picture shows his wife Maria, also at a young age. The lower left photo shows Maria holding their newborn son Andrei Andreevich Markov, Jr. in 1903. Finally, the lower right photo shows a middle-aged Markov, Senior.

The second page of the photo gallery contains two photos. The upper pane shows the older Markovs with young Andrei, Jr. Markov, Sr. is in his early fifties. The lower pane shows the Markovs relaxing in the living room of their home in St. Petersburg.

The third page of the photo gallery contains four photos. The two upper photos again show Markov with his wife Maria, both in older age. In the lower left corner is a picture of

Vladimir, who had both a mathematical commonality as well as a physical resemblance to his older brother. Lastly, the lower right photo shows the Markovs relaxing and reading on a park bench. (We were unable to identify the older woman sitting between the Markovs.)



The final photo shows Markov seated at the desk in his house, with his wife Maria nearby. One can easily imagine that this is where his famous work on chain dependence was conceived, since having retired from the university in 1905, he worked mostly from home.

4. Markov's Mathematical Work

4.1. Forerunners of the Markov chain

After the development of the Markov chain, with the advantage of hindsight, we can identify earlier ideas that are connected to the later idea of a Markov chain. Some of the urn problems studied by Laplace, D. Bernoulli, and Ehrenfests are special cases of Markov chains [31]. Markov chains have a direct relationship to Brownian motion [5]. Random walks, such as the gambler's ruin problem, studied by Bunyakovsky, also have clear connections to Markov chains [6]. Similarly, Bachelier's lesser known study of the stock exchange had ties to Markov's later work [30]. However, most of these earlier studies were either unknown or unrecognized by Markov and were certainly not his motivation for studying Markov chains. His long-time correspondent and fellow mathematician Chuprov pointed out some of these forerunners in a 1910 letter to Markov. In a November 11, 1910 letter to Chuprov, Markov wrote "I most humbly beg you to point out to me those articles of Bohlmann and Bachelier to which you refer. Up to now I had thought that I was the first to dwell on the application of the law of large numbers to dependent variables..." [25]. Chuprov's letter, which prompted this remark acknowledged the connections of Markov's predecessors. After carefully reading the articles mentioned by Chuprov, Markov finds his own work on his chains more general, complete, and correct. In a November 15, 1910 letter, Markov writes "I, of course, have seen Bachelier's article but strongly dislike it. I do not attempt to judge its significance for statistics but with respect to mathematics, it has no importance in my opinion. In any case, it does not contain an extension of Bernoulli's theorem to dependent variables" [25]. Then a few days later, on November 18, he continues "The cases I indicated are NOT included in Bohlmann's cases, but contain them as special cases. There is a huge difference. I am prepared to admit that Bohlmann gave an elegant special formula, but he did not point out even one new (after my article) case of the generalization of Bernoulli's theorem" [25]. Such language hints at Markov's research style and character. He was very exacting, thorough, and complete in his own work and also was not shy about doling out what he deemed rightful criticism of the work of his colleagues. In fact, no colleague was excluded from Markov's judgmental eye. Not even his mentor Chebyshev, whom Markov implicitly accused of plagiarism in 1891 [28].

While the above list of forerunners may not have influenced Markov in his work on chains, we can point to other predecessors to Markov's chains with more confidence and influence. We link together the important papers before Markov's famous 1906 chain paper, creating a "Markov chain gang." In the beginning, in 1738, Jacob Bernoulli's sequences of independent random variables were born with the paper "Ars Conjectandi" [2]. In this paper, Bernoulli proved the weak law of large numbers for sequences of independent binary random variables $x_1, x_2, \ldots, x_n, \ldots$ with probabilities $P(x_i = 1) = p$ and $P(x_i = 0) = q$. Almost a century later in 1837, Simeon Poisson generalized Bernoulli's theorem to include probabilities $P(x_i = 1) = p_i$ and $P(x_i = 0) = q_i$ depending on the trial number i [26]. Thirty years later, in 1867, Pafnuty Chebyshev's paper, "On Mean Values" [8], generalized the weak law of large numbers to any sequence of independent random variables with bounded second moments $E(x_i^2)$. In 1900, Markov, Chebyshev's student, generalized this law for the case where the variances do not exist, but for a certain $\delta > 0$, all the moments $E(|x_i - a_i|^{1+\delta})$ are bounded [14]. This sequence of papers laid the foundation for Markov's study of dependent random variables. However, perhaps the most influential papers on Markov's work on chains were the 1898 and 1902 papers by P. A. Nekrasov. The 1898 paper, "General properties of numerous independent events in connection with approximate calculation of functions of very large numbers," sparked an ongoing and public quarrel between the two men [22]. Nekrasov published a related paper in 1902 [23]. In this paper, Nekrasov erroneously claimed that "independence is a necessary condition for the law of large numbers." Being so precise and methodical, Markov read this paper carefully and reacted strongly. Markov set out to counter his adversary's false claim. He began studying certain types of dependent random variables in order to relax the independent random variable assumption that had been held by each of his predecessors. Markov tried to apply the weak law of large numbers and the central limit theorem to his specific sequences of dependent random variables, aiming to extend Chebyshev's conclusions and expose the error in

Nekrasov's thinking. He studied variables $x_1, x_2, \ldots, x_n, \ldots$ whose dependence on one another quickly lessens as their mutual distance increases. This extension of Chebyshev's theorem to certain types of dependent random variables led to the creation of a new field of research—the theory of Markov chains.

In order to emphasize the role that Nekrasov played in spurring Markov to study chains, we include an entire letter from Markov to Chuprov regarding this subject. It is dated November 6, 1910 [25].

No. 3

(Letter from Markov to Chuprov)

6 November 1910

My dear Alexander Alexandrovich:

Of course I was also surprised by your reference to Bruns whom I consider a negligible quantity.

I can judge all work only from a strictly mathematical point of view and from this viewpoint it is clear to me that neither Bruns nor Nekrasov nor Pearson[1] has done anything worthy of note. You speak about some kinds of most general constructions, but I cannot find these constructions in their work.

Meanwhile I do find highly general theorems from authors whom you have entirely forgotten: A. M. Liapunov[2] and A. A. Markov. The unique service of P. A. Nekrasov, in my opinion, is namely this: he brings out sharply his delusion, shared, I believe, by many, that independence is a necessary condition for the law of large numbers. This circumstance prompted me to explain, in a series of articles, that the law of large numbers and Laplace's formula can apply also to dependent variables[3]. In this way a construction of a highly general character was actually arrived at, which P. A. Nekrasov can not even dream about.

I considered variables connected in a simple chain and from this came the idea of the possibility of extending the limit theorems of the calculus of probability also to a complex chain.

Independence is not required for the application of these theorems, but on the other hand it is necessary to assume existence of certain constant quantities. This existence is already assumed by the theory and therefore it is impossible to deduce this from the theory. And so I will stick to my opinion that your reference to Bruns and Nekrasov is wrong, as long as you do not cite for me their general constructions.

With complete respect,

A. Markov

4.2. Markov's Early Work on Chains

And so the Markov chain was born, although the term did not appear for another 20 years when Bernstein [3] used it for the first time in 1926. The concept of chains first appeared in

Markov's 1906 paper [15], in which he considered chains with only two states, 0 and 1. The journal article, however, was not published until 1907. By that time, Markov had enriched the manuscript by adding two more sections, with the final one presenting the general concept of a chain.

In this first paper, A. A. Markov defined the *simple chain* as "an infinite sequence $x_1, x_2, \ldots, x_k, x_{k+1}, \ldots$, of variables connected in such a way that x_{k+1} for any k is independent of $x_1, x_2, \ldots, x_{k-1}$, in case x_k is known" [15]. Markov called the chain *homogeneous* if the conditional distributions of x_{k+1} given x_k were independent of k. He also considered *complex chains* in which "every number is directly connected not with single but with several preceding numbers" [17].

Most of Markov's works were devoted to simple homogeneous chains. By determining the probability $p_{\alpha,\beta}$ of the event $x_{k+1} = \beta$ given that $x_k = \alpha$, he emphasized, that the probabilities $p_\alpha^{(k)}$ of the events $x_k = \alpha$ are connected by the simple formula

$$p_\beta^{(k+1)} = \sum_\alpha p_\alpha^{(k)} p_{\alpha,\beta}. \tag{1}$$

He also produced the following equalities for the mathematical expectations $a_i = E(x_i)$ and $A_\gamma^{(i)} = E(x_{k+i}|x_k = \gamma)$:

$$a_{k+i} = \sum_\alpha p_\alpha^{(k)} A_\alpha^{(i)}, \quad A_\alpha^{(i)} = \sum_\beta p_{\alpha,\beta} A_\beta^{(i-1)}. \tag{2}$$

In this same 1906 paper, [15], Markov studied the correctness of the weak law of large numbers for homogeneous chains with positive transition probability matrices. From equality (2) he determined that a_{k+i} is located between the smallest $m^{(i)} = \min_\gamma A_\gamma^{(i)}$ and the largest $M^{(i)} = \max_\gamma A_\gamma^{(i)}$ values of $A_\gamma^{(i)}$. Further, these latter numbers are located between $m^{(i-1)}$ and $M^{(i-1)}$. Markov proved that if i increases, the difference $\Delta^{(i)} = M^{(i)} - m^{(i)}$ tends to 0, so the mathematical expectations a_{k+i} and $A_\gamma^{(i)}$ have the same limit a. His elegant proof of this ergodic theorem, as we call it today, deserves to be presented here.

Theorem 4.1. *(Markov, [15]). For a chain with a positive transition matrix all the numbers a_{k+i} and $A_\gamma^{(i)}$ have the same limit, which they differ from by numbers less than $\Delta^{(i)}$. At the same time, $\Delta^{(i)} < CH^i$, where C and H are constants and $0 < H < 1$.*

Proof:

As the sum $\sum_\gamma (p_{\alpha,\gamma} - p_{\beta,\gamma}) = 0$, then grouping its positive and negative terms, we get two equal sums less than 1,

$$h_{\alpha,\beta} = \sum_{\gamma: p_{\alpha,\gamma} > p_{\beta,\gamma}} (p_{\alpha,\gamma} - p_{\beta,\gamma}) = \sum_{\gamma: p_{\alpha,\gamma} < p_{\beta,\gamma}} (p_{\beta,\gamma} - p_{\alpha,\gamma}) < 1.$$

Now changing in the equality by multiplying through by $A_\gamma^{(i-1)}$,

$$A_\alpha^{(i)} - A_\beta^{(i)} = \sum_{\gamma: p_{\alpha,\gamma} > p_{\beta,\gamma}} (p_{\alpha,\gamma} - p_{\beta,\gamma}) A_\gamma^{(i-1)} - \sum_{\gamma: p_{\alpha,\gamma} < p_{\beta,\gamma}} (p_{\beta,\gamma} - p_{\alpha,\gamma}) A_\gamma^{(i-1)}.$$

Bounding some of the numbers $A_\gamma^{(i-1)}$ by the largest of them and others by the smallest, we get the inequality

$$A_\alpha^{(i)} - A_\beta^{(i)} \leqslant M^{(i-1)} \sum_{\gamma: p_{\alpha,\gamma} > p_{\beta,\gamma}} (p_{\alpha,\gamma} - p_{\beta,\gamma}) - m^{(i-1)} \sum_{\gamma: p_{\alpha,\gamma} < p_{\beta,\gamma}} (p_{\beta,\gamma} - p_{\alpha,\gamma}) = h_{\alpha,\beta} \Delta^{(i-1)}.$$

Further, it turns out that $\Delta^{(i)} < H\Delta^{(i-1)}$ and the estimate $\Delta^{(i)} < CH^i$ is correct, where $H = \max_{\alpha,\beta} h_{\alpha,\beta} < 1$, and the constant C is equal to the difference between the maximum and minimum values of the chain. This estimation shows that when i increases indefinitely, $\Delta^{(i)}$ tends to the limit 0. Since a_{k+i} is sandwiched between $m^{(i)}$ and $M^{(i)}$, as i increases, a_{k+i} approaches a limit, in fact, the same limit that $A_\gamma^{(i)}$ approaches.

□

Seneta notes the connection between Markov's H and the ergodicity coefficient $\tau_1(P)$ in an interesting paper [28].

In his 1908 paper [17], Markov dropped the assumption that the transition matrix P be positive and described what we call irreducible chains in the following way: "we consider only those chains $x_1, x_2, \ldots, x_n, \ldots$, where the appearance of some of the numbers $\alpha, \beta, \gamma, \ldots$ does not rule out the possibility of the ultimate appearance of the others" [17]. Today we say that every state is reachable from every other state. Markov produced the following criterion for what would later be called the irreducibility of a homogeneous chain: "the determinant

$$\begin{vmatrix} u & p_{\beta,\alpha} & p_{\gamma,\alpha} & \cdots \\ p_{\alpha,\beta} & v & p_{\gamma,\alpha} & \cdots \\ p_{\alpha,\gamma} & p_{\beta,\gamma} & w & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{3}$$

with variables $u, v, w, \ldots$ cannot be transformed into the product of several determinants of the same type" [17]. A formal definition of the concept of irreducibility would not come until 1912 with Frobenius' famous paper [11]. It appears that we can translate Markov's condition for irreducibility into today's language: "no symmetric permutation (permutation similarity) can transform P into the block form $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$."

4.3. Markov's Later Work on Chains

In his 1906 paper [15], Markov studied the application of the weak law of large numbers to sequences of dependent random variables. He considered a stationary homogeneous chain having two states 0 and 1 with transitional probabilities $p_{1,1} = p_1$, $p_{1,0} = q_1 = 1 - p_1$, $p_{0,1} = p_2$, $p_{0,0} = q_2 = 1 - p_2$, and stationary probabilities $p_1^{(k)} = p$, $p_0^{(k)} = q = 1 - p$, where $p = pp_1 + qp_2$. Having obtained the probabilities of returning from state 1 to 1 in m steps ($R_m = p + q\delta^m$, where $\delta = p_1 - p_2$), he then produced the inequality

$$E(z_1 + z_2 + \cdots + z_n)^2 < Gn, \tag{4}$$

where $z_i = x_i - p$ and G is a constant. Subsequently, applying Chebyshev's inequality, Markov discovered that the weak law of large numbers can be applied to his particular case.

Markov completed this paper by also establishing the possibility of the application of the weak law of large numbers to any homogeneous finite chain with a positive transition matrix. Using the above-mentioned theorem, Markov obtained an inequality similar to (4) and again applied Chebyshev's inequality. In conclusion, Markov emphasized that "the independence of variables does not result in a necessary condition for the validity of the law of large numbers" [15].

In his later papers, [16, 17, 19, 20], Markov studied the applicability of the central limit theorem to various homogeneous chains. For this purpose, he considered the generating function $\Omega(t, z) = \sum_{m,n} P_{m,n} t^m z^n$, where the probability $P_{m,n}$ is defined as $P_{m,n} = P(x_1 + x_2 + \ldots + x_n = m)$. With the help of this generating function, the moments $m_{i,n}(a) = E\left(X_n\right)^i$ of the normalized random variables

$$X_n = \frac{x_1 + x_2 + \cdots + x_n - an}{\sqrt{n}}, \tag{5}$$

can be found easily. This generating function can be expressed as a ratio of two polynomials

$$\Omega(t, z) = \frac{f(t, z)}{F(t, z)}. \tag{6}$$

Moreover, the numerator does not play a significant role in analyzing the behavior of the moments $m_{i,n}(a)$ as $n \to \infty$. In fact, the behavior of the moments is determined by the roots of $F(1, z)$. If some restrictions are placed on the transition matrix, then $F(1, z)$ has a simple root $z = 1$, and the moduli of other roots exceed 1. So, for a properly chosen a, and for any $i = 0, 1, \ldots$, the following limit exists

$$\lim_{n\to\infty} m_{i,n}(a) = \frac{1}{\sqrt{\pi}}\, C^{1/2} \int_{-\infty}^{+\infty} t^i e^{-t^2/2} dt, \quad C = 2\, \frac{\frac{d^2}{du^2}\, F(e^u, e^{-au})|_{u=0}}{\frac{d}{dz} F(1, z)|_{z=1}}. \tag{7}$$

This proves the validity of the central limit theorem.

In his 1908 paper [17], for an irreducible chain satisfying an additional condition, Markov proved that one is a simple eigenvalue of the transition matrix, and the moduli of the other eigenvalues are less than 1. This additional condition is stronger than the condition requiring a chain to be aperiodic, and, as the author himself mentioned, “our conclusions can be extended to many of the cases not considered here” [17]. An interesting paper by Schneider examines Markov’s irreducibility condition and this additional condition in detail [27]. This 1977 paper presents the extent to which Markov intentionally discovered parts of the famous Perron-Frobenius theory of stochastic matrices. Schneider’s conclusion is that Markov’s 1908 work does prove a substantial part, but not all, of the Perron-Frobenius theorem of nonnegative matrices, which Frobenius would prove in entirety four years later in 1912 [11].

Markov proved the central limit theorem for a nonhomogeneous chain with states 0 and 1 under the assumption that all the transition probabilities $P(x_{k+1} = j|x_k = i)$ are equally bounded below by a certain constant $\varepsilon > 0$ [18]. The applied method “means that all the mathematical expectations of the sum powers are divided, according to Newton’s formula, into separate sums, from which the fastest growing with the number of the variables increasing are singled out” [18].

For a sequence of independent random variables, having values 0 and 1, the limiting distribution of X_n, given in (5) is normal with variance $v = pq$. Markov found in [17] that for a simple homogeneous chain, the corresponding variance is given by

$$\nu_\delta = pq\frac{1+\delta}{1-\delta}. \tag{8}$$

Hence, for a simple chain, the equality $v_\delta = v$ holds if and only if $p_{1,1} = p_{0,1} = p$, i.e., when the variables $x_1, x_2, \dots$ are independent. Markov examined whether the same held for other chains with two states [19]. In that 1911 paper he considered the limiting distribution of the variables (5) for a complex homogeneous chain, in which every state is determined not by the lone preceding state but by the two preceding states. He found it is impossible to determine whether the variables $x_1, x_2, \dots$ connected in a complex chain were independent, knowing only their stationary distribution p, q and the limiting variance v_δ of the variables (5).

In 1913, for the 200th anniversary of J. Bernoulli’s publication [2], Markov had the third edition of his textbook [14] published. This edition included his 1907 paper, [16], enriched by the materials from his 1913 paper [21]. In that edition he writes, “Let us finish the article and the whole book with a good example of dependent trials, which approximately can be regarded as a simple chain”. In what has now become the famous first application of Markov chains, A. A. Markov, studied the sequence of 20,000 letters in A. S. Pushkin’s poem “Eugeny Onegin”, discovering that the stationary vowel probability $p = 0.432$, that the probability of a vowel following a vowel is $p_1 = 0.128$, and that the probability of a vowel following a consonant is $p_2 = 0.663$. In the same article, Markov also gave the results of his other tests; he studied the sequence of 100,000 letters in S. T. Aksakov’s novel “The Childhood of Bagrov, the Grandson”. For that novel, the probabilities were $p = 0.449$, $p_1 = 0.552$, and $p_2 = 0.365$. What an enormous amount of tedious calculation! Yet Markov was undaunted by extensive calculations and was very good at them. In fact, his own view on calculating was that “many mathematicians apparently believe that going beyond the field of abstract reasoning into the sphere of effective calculations would be humiliating” [13]. Thankfully, Markov enjoyed entering this sphere. The mathematician Besikovitch remarked that in Markov’s book “great attention is paid to the simplest numerical examples which are discussed in unusual detail. And further, it is hardly possible to find a single mistake in these examples" [4]. Markov also produced his own table for the normal distribution, which he included in the appendix of his book. This table was so accurate that it was used into the 1940s [10].

4.4. Markov's correspondence with Chuprov

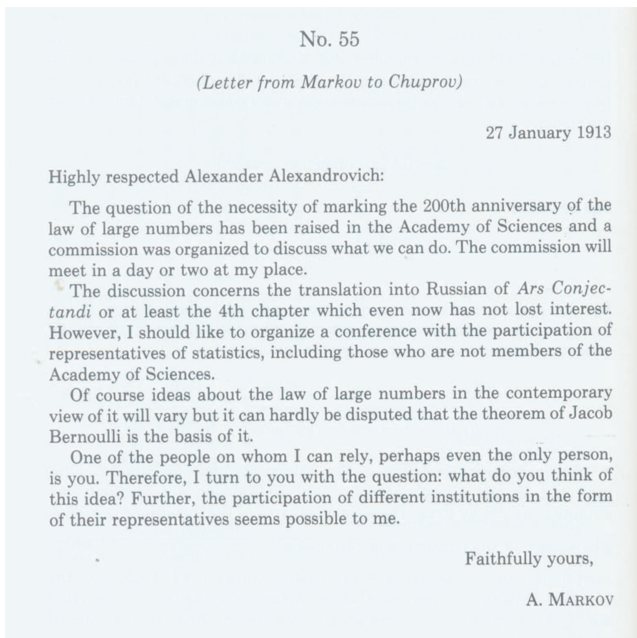
Several of the quotations in the preceding sections came from letters that A. A. Markov wrote to his colleague A. A. Chuprov. The two men maintained an academic correspondence from 1910 to 1917. Over 100 letters from that time have been found. However, many more were lost. In fact, only one letter from Chuprov during the entire year of 1912 was found, and all of Chuprov's letters from 1913 and 1915 were lost. Thus, the conversation that remains from the recovered letters is rather one-sided. Fortunately, Markov's letters are so detailed that readers can easily surmise the gist of the missing Chuprov letters. The Markov-Chuprov communication was almost entirely written, however, we do know that the men phoned and met at least once. For example, in a December 4, 1912 letter Markov tells Chuprov of the new phone just installed in his house and asks Chuprov to phone him to discuss details of the Bernoulli Bicentennial Celebration the two were planning. It is fortunate for us mathematical historians that the two men wrote so often, rather than meeting by phone or in person. The letters leave a wonderful trail of the foundations of modern probability theory as it was laid by two trailblazers.

Both men lived and taught in St. Petersburg. Markov taught at St. Petersburg University, while Chuprov at Petersburg Polytechnic Institute. Both men also wrote textbooks on probability and statistics. Markov's textbook, "Calculus of Probabilities," was more mathematical than Chuprov's 1909 textbook, "Essays on the Theory of Statistics." The correspondence began when, after reading Chuprov's textbook, Markov took issue with the wording of a particular paragraph in the book. As a result, Markov fired off his first note to Chuprov, a brief postcard dated November 2, 1910. Upon reading Chuprov's newly published statistics book, Markov "noticed with astonishment that in the book of A. A. Chuprov, "Essays on the Theory of Statistics", on page 195, P. A. Nekrasov, whose work in recent years represents an abuse of mathematics, is mentioned next to Chebyshev." And here appears the famed enemy Nekrasov once again. He was arguably one of the most influential men in Markov's life. He again seems to be the precipitating force in Markov's life, this time spurring Markov to initiate what later became a very productive and affecting correspondence with A. A. Chuprov. Most of their subsequent letters discuss proper citation and attribution of earlier contributors. They banter about the scientific merit and correctness of the probability and statistical texts of the day. They edit each other's papers and editions of books. This correspondence influenced both men, but we have direct evidence of Chuprov's influence on Markov. In a December 8, 1912 letter Markov wrote to Chuprov saying that "[this] correspondence had some influence on the first chapter of my book." There is also evidence that Chuprov's diplomatic and gentle, persistent persuasion seemed to have eased Markov's negative opinions of statisticians, so much so that near the end of the correspondence we see Markov engaging in his own statistical studies as well as inquiring about the studies of Pearson and Chuprov's students. One other direct outcome of this correspondence was the two men's planning and participation in the

Bicentennial Celebration of the law of large numbers, honoring Jacob Bernoulli's 1713 paper, "Ars Conjectandi." In a January 15, 1913 letter to Chuprov [25], Markov writes:

> do you know: the year 1913 is the two hundredth anniversary of the law of large numbers (*Ars Conjectandi*, 1713), and don't you think that his anniversary should be commemorated in some way or other? Personally, I propose to put out a new edition of my book, substantially expanded. But I am raising the question about a general celebration with the participation of as large a number of people and institutions as possible.

Markov continues with the celebration plans in a January 27, 1913 letter [25]. We include the letter in its entirety below. This letter also reveals the high opinion of Chuprov Markov had developed over the course of their correspondence.



No. 55

(Letter from Markov to Chuprov)

27 January 1913

Highly respected Alexander Alexandrovich:

The question of the necessity of marking the 200th anniversary of the law of large numbers has been raised in the Academy of Sciences and a commission was organized to discuss what we can do. The commission will meet in a day or two at my place.

The discussion concerns the translation into Russian of *Ars Conjectandi* or at least the 4th chapter which even now has not lost interest. However, I should like to organize a conference with the participation of representatives of statistics, including those who are not members of the Academy of Sciences.

Of course ideas about the law of large numbers in the contemporary view of it will vary but it can hardly be disputed that the theorem of Jacob Bernoulli is the basis of it.

One of the people on whom I can rely, perhaps even the only person, is you. Therefore, I turn to you with the question: what do you think of this idea? Further, the participation of different institutions in the form of their representatives seems possible to me.

Faithfully yours,

A. Markov

Chuprov's reply is lost, but we know he agreed to help with the celebration and prompted Markov to reply a few days later with this excerpt from his January 31, 1913 letter [25].

> Your plan—to publish a special collection of articles—requires thorough discussion and dealings with many persons. The organization of an honorary celebration on an international level lends great importance to it, of course, but at the same time introduces considerable complications.
>
> For me it is even unclear whom it would be appropriate to involve in collaboration in the collection. It is clear that in the current year it is impossible to carry out your plan. The initiative for carrying it out should belong to you; I am prepared to assist you, but under the condition that it does not interfere with carrying out the simple celebration which I proposed yesterday to the Academy's commission and which they approved.
>
> In the autumn of this year we intend to organize a general conference devoted to the law of large numbers, in which we count on your participation. Besides you and me, it was proposed to bring in Professor A. V. Vasiliev to this work.

Then it was proposed to translate only the fourth chapter of *Ars Conjectandi*; the translation will be done by the mathematician Ya. V. Uspensky who knows the Latin language well, and it should appear in 1913.

Finally, I propose to do a French translation of the supplementary articles in my book with a short foreword about the law of large numbers, as a publication of the Academy of Sciences.

All of this should be scheduled for 1913 and a portrait of J. Bernoulli will be attached to all publications.

After all this planning in the early part of the year, the actual celebration did not take place until the end of the year on December 1, 1913. One of the three speakers, Vasiliev, must have proposed that all three talks be published in one collection, to which Markov replied in a December 3, 1913 letter to Chuprov [25].

No. 62

(Letter from Markov to Chuprov)

3 December 1913

Highly respected Alexander Alexandrovich:

In order to avoid any misunderstanding I hasten to inform you that I cannot subscribe to A. V. Vasiliev's project to put together a general collection of the three talks. We joined together only temporarily on the basis of mutual independence. From the outside the celebration appears to have turned out fairly successfully. But there is not complete solidarity between us. In your talk statistics stood first and foremost, and applications of the law of large numbers were advanced that seem questionable to me.

By subscribing to them I can only weaken that which for me is particularly dear: the rigor of judgments I permit.

For example, it is very important for me to observe that Poisson did not prove his theorem; moreover, I cannot consider a statement a theorem unless it is established precisely.

Your talk harmonized beautifully with A. V. Vasiliev's talk but in no way with mine. I had to give my talk since the 200th year of a mathematical theorem was being celebrated, but I do not intend to publish it and I do not wish to. The combination of your talk with A. V. Vasiliev's talk without mine is quite possible, as is clear, by the way, from P. Yushkevich's article in the newspaper *The Day* (of 2 December, No. 327); and I suggest it be done.

Sincerely yours,

A. Markov

This letter is again very typical of Markov's style, his exacting attention to detail, precision, and correctness as well as his bluntness when it comes to insulting colleagues. Understandably, this letter resulted in a long break in the Markov–Chuprov correspondence (No. 62 is dated December 3, 1913 while No. 63 is dated February 3, 1915) [29]. The book, "The Correspondence between A. A. Markov and A. A. Chuprov on the Theory of Probability and Mathematical Statistics" contains the complete commemorative addresses given by both Markov and Chuprov in December 1913 [25].

5. Conclusion

Markov proved that the independence of random variables was not a necessary condition for the validity of the weak law of large numbers and the central limit theorem. He introduced a new sequence of dependent variable, called a chain, as well as a few basic concepts of chains such as transition probabilities, irreducibility and stationarity. His ideas were taken up and developed further by scientists around the world and now the theory of Markov Chains is one of the most powerful theories for analyzing various phenomena of the world.

Acknowledgement. The authors would like to thank B. A. Sevastianov, a member of the Russian Academy of Sciences, and D. B. Gnedenko, professor at Moscow State University, for their help in preparation of this article. The authors also thank Andrey Bovykin of the Laboratory of Mathematical Logic, Petersburg Department of Steklov Institute of Mathematics for recommending many of the pictures of Markov with his family and for obtaining the necessary permissions to include the pictures in this article. The authors also thank the referees for their many valuable comments, which greatly improved the presentation of the paper.

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