THE INTRODUCTION OF THE SYMBOL π IN 1706

Since antiquity, the calculation of the value of π has been a challenge for people from many parts of the world. The constant π is defined as the ratio of the circumference or periphery, $\pi\,\epsilon\,\rho\,\iota\,\phi\,\acute{\epsilon}\,\rho\,\epsilon\,\iota\,\alpha$ in Greek, p of a circle to its diameter d

$$\pi \;=\; p\,/\,d$$

Johann Lambert proved that π is irrational in 1768 and in 1882 Ferdinand von Lindemann showed that π is transcendental, i.e., not the root of any algebraic equation with rational coefficients. Some milestones in the evaluation of the decimal digits of π would include

Location	Name	Year	π Digits
Babylon	Egypt (Middle Kingdom)		2
Sicily	Archimedes	-250	3
India	Aryabhata	400	4
China	Liu Hui	263	5
China	Zu Chongzhi	480	7
India	Madhava	1400	13
Persia	Kashani	1430	16
Germany	Ludolf van Ceulen	1610	35
London	John Machin	1706	100
Slovenia	Jurij Vega	1794	136
Hamburg	Johann Dase	1844	200
Durham	William Shanks	1873	527
England	D F Ferguson	1947	808
Maryland	Daniel Shanks	1961	10^{6}
Brooklyn	G & D Chudnovsky	1989	10^{9}
Tokyo	Yasumasa Kanada	2005	10^{12}

In 1706 William Jones published a work entitled "*Synopsis Palmariorum Matheseos*, or, A New Introduction to the Mathematics, Containing the Principles of Arithmetic and Geometry". From Anglesey in Wales, he was the first to use π with its present meaning. Page 243 has

> "There are various other ways of finding the lengths, or areas of particular curve lines, or planes, which may very much facilitate the practice; as for instance, in the circle, the diameter is to circumference as 1 to
>
> $$\left(\frac{16}{5} - \frac{4}{239}\right) - \frac{1}{3}\left(\frac{16}{5^3} - \frac{4}{239^3}\right) + \frac{1}{5}\left(\frac{16}{5^5} - \frac{4}{239^5}\right) - \&c. \;=$$
>
> 3.14159, &c. $= \pi$. This series (among others for the same purpose, and drawn from the same principle) I received from the excellent analyst, and my much esteemed friend Mr. John Machin; and by means thereof, van Ceulen's number may be examined with all desireable ease and dispatch."

A derivation of the expression of John Machin for π from an addition formula for arc tangents is provided below. The demonstration relies upon a series for the arctangent function conceived by the Kerala School of Mathematics in India during the 15th century and established independently by James Gregrory from Scotland in 1671 and by Gottfried Wilhelm Leibniz from Germany in 1673.

A derivation of Machin's formula for π can begin with the addition formula for tangents

$$\tan(A+B) \;=\; \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

so that writing $x = \tan A$ and $y = \tan B$ leads to an addition formula for arctangents

$$\tan^{-1} x + \tan^{-1} y \;=\; \tan^{-1} \frac{x+y}{1-xy}$$

that is, in particular when $x = y = p/q$, there results a duplication formula for arctangents

$$2\,\tan^{-1}\frac{p}{q} \;=\; \tan^{-1}\frac{2p/q}{1-p^2/q^2} \;=\; \tan^{-1}\frac{2pq}{q^2-p^2}\,.$$

Observing, perhaps, that the angle $4\,\tan^{-1}(1/5)$ is close to half a right angle, consider

$$4\,\tan^{-1}\frac{1}{5} \;=\; 2\,\tan^{-1}\frac{10}{25-1} \;=\; 2\,\tan^{-1}\frac{5}{12}$$

which, upon a further use of the duplication formula with $p = 5$ and $q = 12$, yields

$$2\,\tan^{-1}\frac{5}{12} \;=\; \tan^{-1}\frac{120}{144-25} \;=\; \tan^{-1}\frac{120}{119}$$

leading to a result close to $\tan^{-1}(1) = \pi/4$ that invites a study of the difference; thus

$$4\,\tan^{-1}\frac{1}{5} \;=\; \tan^{-1}\frac{120}{119}\,.$$

Noting that $\tan^{-1}(-y) = -\tan^{-1}(y)$ is an odd function, a substitution of $x = 120/119$ and $y = -1$ into the above addition formula for arctangents gives

$$\tan^{-1}\frac{120}{119} - \tan^{-1}(1) \;=\; \tan^{-1}\frac{120/119 - 1}{1 + 120/119} \;=\; \tan^{-1}\frac{1}{239}$$

completing a proof of John Machin's formula dated 1706, recalling that $\tan^{-1}(1) = \pi/4$

$$\pi \;=\; 16\,\tan^{-1}\frac{1}{5} \;-\; 4\,\tan^{-1}\frac{1}{239}\,.$$

Machin evaluated π to 100 decimal digits using the expansion of Gregory in his formula

$$\pi \;=\; 16\left(\frac{1}{5} - \frac{5^{-3}}{3} + \frac{5^{-5}}{5} - \frac{5^{-7}}{7} + \frac{5^{-9}}{9} - \cdots\right) - 4\left(\frac{1}{239} - \frac{239^{-3}}{3} + \cdots\right).$$

Entering the following as a Google search string (or as an argument for `calc` of UNIX)

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16/5 - 16/5^3/3 + 16/5^5/5 - 16/5^7/7 + 16/5^9/9 - 4/239 + 4/239^3/3
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provides eight decimal digits of Archimedes' ratio (3.1415926) or more with additional terms. The first 61 digits of π are the following, the next 10 digits being $\ldots 5923078164$

$$\pi = 3.1415926535\ 8979323846\ 2643383279\ 5028841971\ 6939937510\ 5820974944\ \ldots\,.$$