An ordinary differential equation involves a relation between an independent variable x, a function y of x, and its derivatives; that is, a relation between

$$x, \quad y, \quad \frac{dy}{dx}, \quad \frac{dy^2}{dx^2}, \quad \ldots .$$

A first order ordinary differential equation relates the variables x, y and dy/dx. Section 10.1 of "Calculus" by Howard Anton discusses differential equations.

An 'initial value problem' refers to a first order ODE where the value of y is given at a particular value of x. The constant of integration appearing in the solution of the ODE may be determined from the initial value of y provided. A sample solution of an initial value problem is given overleaf.

Answer the following in relation to first order separable ordinary differential equations.

1. Confirm that $y = 2x^4 + 3\cos x - 1$ solves the initial value problem

$$\frac{dy}{dx} = 8x^3 - 3\sin x\,, \quad y(0) = 2\,.$$

2. $$\frac{dy}{dx} = \frac{y}{x}$$

3. $$\frac{\sqrt{1+x^2}}{1+y}\frac{dy}{dx} = -x$$

4. $$(1+y^2)\frac{dy}{dx} = e^x\, y$$

5. $$e^{-y}\sin x - \frac{dy}{dx}\cos^2 x = 0$$

6. $$\frac{dy}{dx} = \frac{y^2 - y}{\sin x}$$

7. $$\frac{dy}{dt} = \frac{2t+1}{2y-2}, \quad y(0) = -1\,.$$

A sample solution uses the method of separation of variables to solve the following initial value problem

$$\frac{dy}{dx} = \frac{4x^2}{y + \cos y}\,, \qquad y(1) = \pi\,.$$

The variables x and y are separated before integration

$$\begin{aligned}
(y + \cos y)\ dy &= 4x^2\, dx \\
\int (y + \cos y)\ dy &= \int 4x^2\, dx \\
\frac{1}{2}\, y^2 + \sin y &= \frac{4}{3}\, x^3 + C \\
3y^2 + 6 \sin y &= 8x^3 + 6C
\end{aligned}$$

Fractions have been cleared through multiplication by 6. The initial condition $y(1) = \pi$ when $x = 1$ is used to evaluate the constant of integration C,

$$\begin{aligned}
3\pi^2 + 6 \sin \pi &= 8 + 6C \\
6C &= 3\pi^2 - 8\,.
\end{aligned}$$

If possible, the answer should involve explicitly solving the equation for the variable y in terms of x. Here, an explicit solution is not possible, and so the answer is given in implicit form

$$3y^2 + 6 \sin y = 8x^3 + 3\pi^2 - 8\,.$$

A numerical solution of this equation could be used to find y as a function of the variable x.